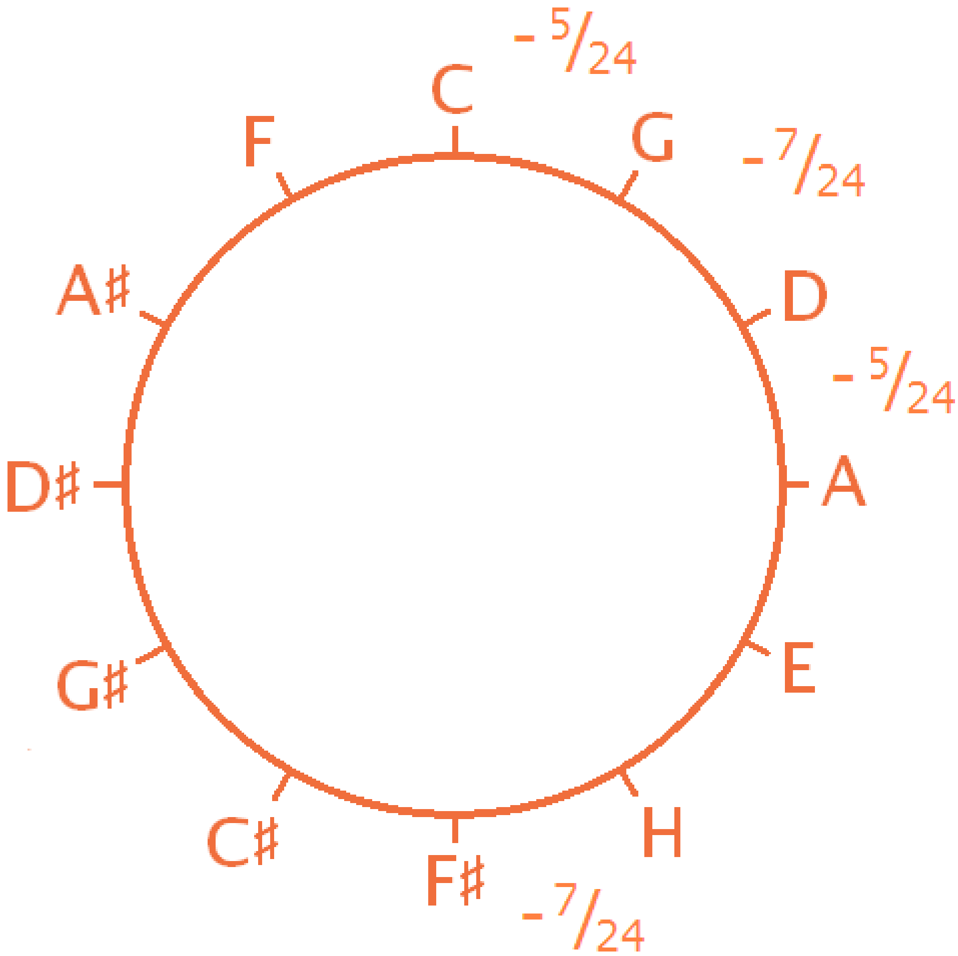
Tuning Werckmeister III: A Thought Experiment

Technical Report

Bjarne Pagh Byrnak



ABSTRACT

An attempt is made to interprete Andreas Werckmeister's definition of the keyboard temperament known as Werckmeister III in a historically informed manner. It is assumed that the first tuners to tune that temperament did not want to change their meantone-oriented tuning habits more than necessary. This report is a feasibility study in which three variants of Werckmeister III are derived from the assumption that the harmonious quality of the triad C–E–G was considered to be more important than strictly equal-sized tempered 5ths. Werckmeister IV and V are considered briefly as a test of methodology. The primary reference is Werckmeister's *Musicalische Temperatur* (1691). The analysis is detailed but avoids mathematics as far as possible.

PREFACE

The author found several years ago that four of the six tempered 5ths in the well-known Bach temperament proposed by John Barnes could be tuned as to resemble those in $^2/_9$ syntonic comma meantone temperament, with seemingly positive implications for its ability to present Bach's music.

Rather than continuing to experiment blindly, I decided to analyse the historical background for tuning tempered 5ths to different sizes in cases where standard modern practice is to tune them equal-sized. The plan was to scrutinize essentially each and every detail in a lengthy report, then decide later whether or not a briefer paper could be destilled from it and possibly published.

It gradually became clear that the task was bigger than expected. To extend the analysis to cover historical literature beyond Werckmeister's *Musicalische Temperatur*, as would be appropriate in academic work, and to include other historical temperaments, would have lasted decades.

At the time when this report was about half finished, Bradley Lehman's twin "Rosetta" papers arrived on the scene. While Werckmeister III will always be regarded as an important historical temperament, it is likely to be used less often for Bach's music in the future, and not much would be gained by pursuing the subject further.

So, the report is presented here as is.

CONTENTS

Technical Remarks

Many cross references are clickable in the on-line version despite not being highlighted – and so are the entries in the table-of-contents on the previous page. In the PDF document, ALT+Left Arrow should bring you back to the pre-click position. Internet URLs in the References will activate your browser if you click them. This is meaningful only if the destination home pages still exist and still contain the relevant information – which is far from certain after some years.

1. Introduction

The historical keyboard temperament known as Werckmeister III is defined in a table on top of p.78 in the *Musicalische Temperatur*, the well-known book published by Andreas Werckmeister in 1691. The temperament is sometimes called “Correct Temperament No. 1” or “Werckmeister III Correct Temperament No. 1.” Werckmeister called it “Temperatur Num. 3” in the text on p.57 of the *Musicalische Temperatur*, “Die Erste Art Num. 3” in the table’s heading, and “Num. III” on the engraving that depicts the string lengths graphically (the *Kupferblatt*).

The table is divided by a vertical line into two parts.

- The left-hand part specifies the sizes of the keyboard’s twelve 5ths. Of those, C–G, G–D, D–A, and B–F♯ are tempered ¼ commatis flat. The rest are pure.
- The right-hand part shows the sizes of the keyboard’s twelve major 3rds. They are all sharp:
  ¼ commatis for F–A and C–E;
  ²⁄₄ commatis for B♭–D, G–B, and D–F♯;
  ¾ commatis for E♭–G, A–C♯, E–G♯, and B–D♯; and
  ⁴⁄₄ commatis for F♯–B♭, C♯–F, and G♯–C.

Three more temperaments of basically the same kind are presented in the same book:

- Werckmeister IV Correct Temperament No. 2, which is defined in a similar table and for which a list of string lengths is also provided;
- Werckmeister V, defined only in table form; and
- the “Septenario” Werckmeister VI, defined solely by two string length lists in which the prime factor 7 is used ingeniously. (It is *not* a ⅐ comma temperament as is sometimes stated in the modern literature; in fact, no kind of comma is used in the definition.)

The tables defining Werckmeister III and IV had been published before, in Werckmeister’s *Orgel-Probe* (1681). The present report is primarily about Werckmeister III, with the *Musicalische Temperatur* as the primary source. The other temperaments will be referred to occasionally.

Werckmeister’s temperaments allowed players to modulate through all major and minor keys without encountering strongly dissonating intervals. An instrument so tuned was said to be well tempered (“wol temperirt” on the title page of the *Musicalische Temperatur*, “wohl temperiret” on p.61; while Bach’s spelling in *Das Wohltemperirte Klavier* from 1722 shows some resemblance to both, recent research suggests that his temperament was not identical to any of Werckmeister’s, although it largely followed the principles outlined in the *Musicalische Temperatur*, cf. the Postscript on page 33). Today, a temperament of this kind is sometimes called a well temperament. Werckmeister called it a correct temperament (“richtige Temperatur”) in order to distinguish it from the “unrichtige Temperatur,” now called meantone temperament, in which some keys with many accidentals were more or less unplayable because of dissonances. The playable keys in meantone temperament could be made to sound with a high degree of harmonious quality.

Most keys in Werckmeister’s temperaments sounded with reduced quality compared to meantone temperament. Some organ builders continued to use meantone temperament, prompting a remark by Werckmeister about bad habitual thinking (“böse Gewohnheiten” -- p.76 in the *Musicalische Temperatur* – Quote 1). Others with more benign habitual thinking agreed with Werckmeister and used his temperaments. We shall assume throughout this report that these tuners would want the best keys in Werckmeister III to sound with a quality comparable to meantone temperament, and that they would read Werckmeister’s table from that viewpoint.

Modern tuners tend to tune Werckmeister III in one and the same way. They do so because they rely on one and the same interpretation of Werckmeister's table. Let us call it the standard modern interpretation. The logic behind it is very simple:

> Since each of the tempered 5ths is specified to be ¼ commatis flat, it must be theoretically correct to tune them strictly equal-sized. Mathematics tells us that the only way to do so is to flatten each of them by one-quarter of a ditonic comma, or 5.865 cents. To calulate cent values for the standard modern interpretation is straightforward. They are listed in Table 4 on page 39.

Equal-sized tempered 5ths are a natural choice in the absence of reasons for doing otherwise, and one should not postulate that the standard modern interpretation is wrong. It is a valid first approximation if one defines that concept to mean a usable but not necessarily accurate result based on relatively little analysis. Evidence in favour of it can be found in the *Musicalische Temperatur* if one reads the book selectively (see page 15 of this report). However, it is certainly not the only possible way to interprete Werckmeister's table, and it is not necessarily the one that suits the music best.

The purpose of this report is to try and extend the analysis down to the next deeper level, and to re-interprete the specification table in an attempt to get closer to the way Werckmeister III was tuned historically. (The result could be called a second approximation. It is, of course, valid only if the extended analysis is correct. The »if« is bigger than it looks, as the opportunity for missteps is extended too.) As a starting point, let us identify two apparent weaknesses in the logic behind the standard modern interpretation.

- The standard modern interpretation neglects the part of Werckmeister's table in which the sizes of the major 3rds are described. It will be argued in Section 2 that the major 3rds were considered at the time to be important.
- The standard modern interpretation relies heavily on the ditonic comma. Matematicians may argue that this has to be so, but Werckmeister expressed himself differently. The ditonic comma is fully covered in the *Musicalische Temperatur*; it is, however, treated as a side issue of mostly theoretical and historical interest. There is no indication in the book that Werckmeister wanted it to be used for practical tuning purpose. Werckmeister's presentation of the ditonic comma is analysed in Section 4.

The first thing we shall do in an attempt to remedy these shortcomings is to adopt a working hypothesis. It will be split up into three parts for the sake of clarity:

- The first tuners to tune Werckmeister III had acquired their tuning experience by tuning meantone temperament.
- These tuners, when tuning meantone temperament, did their best to make the good triads to sound with the desired quality.
- They did the same when tuning Werckmeister III.

(It is not assumed that they tuned triads directly. It *is* assumed that they did not adhere strictly to mathematical theory when tuning.) The sizes of certain major 3rds will be considered in the analysis along with those of the 5ths. One of the Variants of Werckmeister III to be presented in the following will be derived mathematically in a way that makes use of both the ditonic and the syntonic comma, as a more direct way has so far not been found. Two other Variants materialise in a natural way without any comma being used.

The usual modern quantitative methods will be used to approximate certain aspects of old subjective tuning techniques, so that cent values can be calculated. The methodology will be relatively simple:

- The good major triads in Werckmeister III will be identified (this is an easy task, as there are only two of them). Some of the involved intervals will be selected; the sizes of these intervals, as specified in Werckmeister's table, will be included in the analysis.
- Two intervals will be considered at a time. Their entries in the table will be compared and quantitative information extracted. With this pairwise approach, one does not have to assume an *a priori* type or size for the commatis (cf. Section 3).
- Tempered intervals will be allowed to deviate slightly from their specified sizes. Pure intervals will not.

The Variants will be allowed to have slightly unequal-sized tempered 5ths. As for the permissibility of this, meantone temperament is thought to have been tuned with slightly unequal-sized 5ths quite often. (Selected aspects of old subjective tuning techniques are outlined but not analysed in Section 5). Perhaps more significantly, Werckmeister endorsed tuning Werckmeister IV according to a string length list (p.80 in the *Musicalische Temperatur*) in which the seven tempered 5ths in that temperament are flat and sharp by seven different amounts ranging from 5.314 cents to 8.635 cents; all of these values, which are listed in Table 8, represent ⅓ commatis according to Werckmeister's table of specifications.

The analysis rests in part on assumptions that are not based on evidence in the *Musicalische Temperatur*. Werckmeister did not say that tuners should emphasise the quality of certain triads, or that certain specifications should be given priority or should be interpreted pairwise — or whether the tables should be interpreted at all; they may have appeared self-explanatory to tuners of the time. Modern theorists have to interprete the tables before cent values can be calculated, for reasons explained in Section 3. Interpretation might have been easier if Werckmeister had revealed to us the full technical details of tuning as it was practiced at the time. This was clearly not the purpose of his book; furthermore, how to tune a 5th to the right size without beat counting was probably more easily demonstrated on the instrument than explained in words. The few things the book tells us are certainly valuable: Tuning started from C and proceeded 5ths-up-octaves-down around the circle to end an octave higher than it started; and certain major 3rds would be checked and corrections made as needed until they sounded tolerably. However, such information does not tell us how a particular triad sounded. Some remarks in Werckmeister's foreword indicate that he may have had his reasons for being frugal with tuning details:

- Some readers of the *Orgel-Probe* had objected that Werckmeister had revealed "arts and secrets" of organ building in that book (Quote 2: "... daß ich der Orgelmacher künste und Heimlichkeiten zum Theil offenbahret" -- third page of the *Vorrede* of the *Musicalische Temperatur*). Similarly, the finer details of organ tuning may have been considered a business secret by some organ builders.
- Werckmeister saw no need for teaching experienced musicians how to tune a keyboard instrument, for they knew already how to give and to take (Quote 3: "Ein wohlgeübter Musicus practicus weiß schon selbst zu geben und zu nehmen" -- two last pages of the *Vorrede*). To give and to take means to temper a pure interval into a sharp or a flat interval. This can be seen from a remark on p.30 in the *Musicalische Temperatur* where readers are urged to verify for themselves that a 4th or a 5th becomes dissonant if one gives it a comma or takes a comma away from it, whereas a major 3rd sounds more acceptable (see Quote 25 on page 21 of this report).

Because of this scarcity of direct historical evidence, the present work should be regarded as a first attempt of a »what if« thought experiment.

Notation. Brief in-line references are used in place of footnotes. In particular, »-- p.78« means »on page 78 in the German text of any facsimile edition of the *Musicalische Temperatur*.« Quotes from that book are written with a modern typeface, a modern comma, and a colon for string length ratios.

Novice tuners who are familiar with the basic facts of meantone and well temperament should be able to read this report without difficulty. Owen Jorgensen's big *Tuning* book (1991) and Mark Lindley's articles in the *New Grove* (2001a-b) are well suited as introductions to the subject.

2. The Importance of the Keyboard's Twelve Major 3rds

It will be argued in this section that Werckmeister considered the sizes of the major 3rds to be about as important as those of the 5ths. The standard modern interpretation of Werckmeister III treats the major 3rds as if they were less important than the 5ths. The standard modern interpretation is not based on analysis of the major 3rds in Werckmeister's table, but simply assumes that they get reasonable sizes automatically when the tempered 5ths are tuned strictly equal-sized. Most of the major 3rds acutally do so — except a few that happen to be frequently used.

Consider the major 3rds C–E and F–A in the standard modern interpretation. Both are sharp by an amount very close to ⅙ ditonic comma. This is roughly thirty percent less than the ¼ commatis specified in Werckmeister's table. Compare now with G–B. The standard modern interpretation makes this major 3rd sharp by an amount very close to $\frac{5}{12}$ of a ditonic comma, that is, two-and-a-half times as sharp as C–E. It should be only twice as sharp as C–E according to the table. It looks as if the standard modern interpretation exaggerates the key colour contrast when the player modulates from C major to G major or vice versa. The contrast between F major and B♭ major is similarly exaggerated.

The same is true, but to a much lesser extent, in absolute terms: G–B is specified to be ¼ commatis sharper than C–E. Tuners who were grown up with meantone temperament are likely to have read ¼ commatis as ¼ syntonic comma; the standard modern interpretation takes it to mean the slightly larger amount of ¼ ditonic comma.

The key colour contrasts in Werckmeister III must have sounded unusually strong in the ears of most tuners in the 17th century, even without such exaggeration. It is hard to see why they should have wanted to amplify the contrasts between G major and C major and between B♭ major and F major beyond specifications. If anything, it is easier to imagine them wondering if these contrasts could be adjusted downwards.

That Werckmeister assigned some importance to the major 3rds can be seen from the amount of space he devoted to them in his text. Here is an example: Werckmeister described equal temperament as one in which all 5ths are tempered $\frac{1}{12}$ commatis, the major 3rds ⅔, and the minor 3rds ¾ of a commatis (in the *Musicalische Paradoxal-Discourse*, 1707, p.110, according to Norrback, 2002, p.76). The first clause — that all 5ths are $\frac{1}{12}$ commatis flat — would have been sufficient, as it defines the temperament completely; but Werckmeister seems to have considered it good practice to supply the sizes of the major 3rds, and in a few cases even the minor 3rds, along with those of the 5ths. More examples can be found in the *Musicalische Temperatur*:

- Meantone temperament is described several times as one in which the 5ths are ¼ commatis flat and the major 3rds are pure. Again, the sizes of the 5ths would have been sufficient.
- Each specification table contains the sizes of twelve 5ths and the sizes of twelve major 3rds, as already mentioned.
- The 5ths in Werckmeister III are discussed in detail in the book's *Kapitel XXI*. A similar amount of space — about one page — is devoted in *Kapitel XXII* to an equally detailed discussion of the major 3rds.

The 5ths and the major 3rds are given »equal coverage« in all of the above cases, indicating that Werckmeister considered them to be about equally important. A few counterexamples exist; they do not qualify as counterevidence. Werckmeister demonstrates in his *Kapitel XIX* that the "spurious view held by the old as well as some young musicians and organ builders" — that all 5ths should be ¼ commatis flat — causes certain intervals to be 2 whole commata out of tune. He spends two pages on a monochord-oriented guided tour around the circle-of-5ths in order to expose this common mistake for the eye (...diesen allgemeinen Irrthum vor Augen stellen –p.53; Quote 4). It takes only ten lines to establish that certain major 3rds get "entirely useless" ("gantz unbrauchbar" --p.55); however, to point this out is obviously the purpose of the whole *Kapitel*. The 5ths but not the major 3rds of the "Septenario" temperament are discussed in the book's *Kapitel XXVII*; Werckmeister compensates by considering major 3rds in more generality in his *Kapitel XXVIII*.

That all major 3rds are be sharp in a correct temperament is stated explicitely several times in the *Musicalische Temperatur*. Their relevance for key colour characteristics is hinted at only once: The "somewhat harder" 3rds C♯–F, G♯–c and F♯–B♭ in Werckmeister IV are said to be "quite pleasant [when sounding together with other intervals as to make a full chord], and [also quite pleasant] in relation to change in the harmony, as a variation will be sensed"

> (Quote 5.) "Die Tertien Cis–F, Gis–c und Fis–B[♭] sind zwar etwas härter, aber in vollem concent, und Veränderung der Harmonia gantz angenehm, denn der Sensus bekommt eine variation" -- p.80.

Werckmeister's specification tables were meant as practical tuning guides to have available before one's eyes while tuning ("... die Tabellen ... welche in praxi, wenn man stimmen will, vor die Augen genommen werden" -- p.77; Quote 6). We must assume that Werckmeister included the major 3rds in his tables because he wanted his readers to look at them. Tuners who were interested in the quality of the good triads are likely to have done so.

The frequently used triad C–E–G is one of the two best major triads in Werckmeister III. All of its intervals are tempered, which means that its sound depends to the fullest extent on the tuner's personal tuning style. Werckmeister's table says that C–G is ¼ commatis flat and that C–E is ¼ commatis sharp. We shall assume in the following that tuners in the 17th century would notice these two pieces of information and consider both of them to be important.

A few aspects of Werckmeister's presentation of his temperaments will have to be examined first. A peculiarity in his usage of the commatis will be described in the next Section, as it is probably best to understand it before using the tables.

3. Theorist's Dilemma: Which Type of Comma?

The specification table for Werckmeister III expresses the sizes of all tempered intervals in terms of the commatis. The size of the commatis is not specified. Can it be inferred from the table? An attempt to do so leads to a contradiction, as follows.

The octave has twelve tones. Since one is customarily tuned to some pitch standard, only eleven intervals must be specified in order to define a temperament. Werckmeister's table specifies twenty-four. In modern technical jargon, the table is overdetermined. It is also inconsistent: If we try to infer the size of the commatis from only part of the specifications, the result may depend on which specifications we choose.

If we consider only the specifications for the 5ths, we find that the commatis must be ditonic (23.460 cents). This is because the table says that each of the four tempered 5ths is ¼ commatis flat; mathematics requires the sum of these four amounts to equal one ditonic comma.

If we include the specified size of just one major 3rd — for instance, C–E — into the analysis along with the 5ths, we find instead that the commatis must be syntonic (21.506 cents).

> Proof.
> Let x be the size of the commatis, and let S be the syntonic comma, both measured in cents. The table says that C–E is $\frac{1}{4}$ commatis sharp, that is, $\frac{1}{4}x$. The table also says that the 5ths C–G, G–D, D–A, and A–E are flat by the amounts of $\frac{1}{4}x$, $\frac{1}{4}x$, $\frac{1}{4}x$, and 0, respectively (the zero is merely another way of saying that A–E is pure). As is well known, the amount by which C–E is sharp equals one syntonic comma minus the sum of those four amounts. Thus $\frac{1}{4}x = S - \frac{3}{4}x$, from which $x = S$.

The two results contradict one another, as the commatis cannot be ditonic and syntonic at the same time.

The phenomenon is not difficult to understand. Werckmeister observes in the *Musicalische Temperatur* that the excessus obtained when going through all 5ths equals an 81 : 80 comma plus an additional "small differens." The syntonic comma enlarged with the differens is identical to what we now call the ditonic (or Pythogorean) comma. Using an enlarged comma to flatten the four tempered 5ths makes those 5ths flatter than they would have been without enlargement. Flatter 5ths in turn affect the sizes of most major 3rds; but this fact is nowhere mentioned in the *Musicalische Temperatur*. Instead, the sizes of the major 3rds are calculated from those of the 5ths as if the commatis were syntonic.

How, then, would a practical tuner in the 17th century treat the differens? Either judiciously (one might guess) as to make it interfere as little as possible with the quality of the good keys — or, perhaps more likely, not at all. The ditonic comma comes into play automatically in any temperament in which twelve playable 5ths form a closed circle, even if the tuner does not care about it. A proof of this very elementary fact is given on page 12.

Werckmeister's attitude to the ditonic comma in connection with correct temperament can be inferred from his presentation of it. The full story can be found in Section 4. A few peripheral but content-rich comments and remarks in the *Musicalische Temperatur* will be considered here. On the theoretical side, Werckmeister observes that the differens is a small quantity — about the width of a thin line engraved with the compass on a diagram such as the *Kupferblatt*. Werckmeister comments: "From this we learn that not everything in music can be experienced and comprehended with the senses or by using the compass, but it must be made certain by good old-fashioned calculations"

> (Quote 7.) "... darum sehen wir, daß wir in Musicis nicht alle Dinge durch den Sensum, auch nicht allemal durch den Circinum erfahren und begreiffen können, sondern es muß durch die Rechnung, welches die Alten rationem nennen, gewiß gemachet werden" -- p.66.

In other words, theorists who wanted to understand the differens had to calculate it accurately. In addition, Werckmeister urged his readers to study his temperaments on the monochord. This process was, at least in part, visual: “Yet the monochord must be taken before the eyes, ... and however clear the [verbal] description might be, nothing helps as well as looking at the monochord”

> (Quote 8.) “Jedoch muß das Monochordum vor die Augen genommen werden, ... und wenn die Beschreibung noch so deutlich wäre, so kan doch nichts helffen, als die Betrachtung des Monochordi” -- p.53.

Werckmeister observes (--pp.62-63) that the two quantities 81 : 80 and 2048 : 2025 differ in size by one differens, and that an attempt to engrave both would cause one thin line to touch another, which would not be useful but would just cause confusion:

> (Quote 9.) “... dann [2048 : 2025] darneben ein comma zu pflantzen, hätte eine subtile Linie bey den andern anstreichend herauff müssen, und wäre nur nodum in scirpo quære: und verursachete nur confusion” -- p.63.

The Latin allegory for uselessness means searching for knees in reed (of a species without knees). Later on the same page Werckmeister continues: “This differens causes some difficulties on the monochord, in that it is easily overlooked; one must consider carefully how this very small differens might be included [in the engraving]”

> (Quote 10.) “Diese differens machet einige difficultäten in dem Monochordo, also daß man es gar leicht versehen kan, und muß wohl observieret werden, wie man heraus kommen und diese sehr kleine differens eingetheilet werden möge” -- p.63.

In short, monochord users should decide case by case if and how the differens could be included. In situations where it would do more harm than good, it should be omitted. Werckmeister himself appears to have omitted the differens from his *Kupferblatt*.

The 5ths in ¼ syntonic comma meantone temperament are discussed in Werckmeister’s *Kapitel XIX*. The discussion is based on ¼ commatis and multiples thereof, and the commatis is obviously syntonic. The 5ths in Werckmeister III are similarly discussed in his *Kapitel XXI*. That discussion, too, is based on ¼ commatis and multiples thereof. There is nothing to suggest that the type of commatis has changed between *Kapitel XIX* and *Kapitel XXI*. Both discussions are monochord-oriented; readers are, in fact, urged have the *Kupferblatt* before their eyes while reading. Thus, monochord users were allowed to think in terms of the syntonic comma while familiarising themselves with Werckmeister III.

What about practical tuners? The fact is that the *Musicalische Temperatur* says nothing about what to do with the differens while tuning an instrument. The picture is now taking shape:

- Theorists had to calculate the differens carefully and accurately.
- Monochord users could include or omit the differens depending on the situation.
- In practical tuning, the differens would ordinarily be disregarded.

In the *Musicalische Temperatur*, the differens and the ditonic comma are treated as theoretical concepts with some relevance for monochord work but without implications for practical tuning. Everything in the book supports the view that Werckmeister wanted his readers to tune his temperaments while relying on experience they had acquired by tuning meantone temperament. Their experience was based on tuning by subjective judgment with emphasis on the quality of the good triads and with the syntonic comma as an underlying concept.

As for the very elementary fact mentioned above, here are the details:

Infrequently Asked Question.

Can a correct temperament be tuned correctly if the tuner does not use the ditonic comma?

The Answer is affirmative:

The amounts by which the twelve 5ths are tempered will add up to minus one ditonic comma automatically.

Evidence by Example (without mathematics).

Let a correct temperament be defined by a string length list. We shall demonstrate that the sum of the amounts by which the twelve 5ths are tempered does not change if the string length for one tone is modified. The list on p.80 in the *Musicalische Temperatur* is useful as an example, because it contains a printing error. The uncorrected list reads as follows, in Werckmeister's notation but with B written as B[♭] to prevent misunderstandings.

C	120,	Cis	114 ½,	D	107 ⅕,	Dis	101 ⅕,	E	95 ⅗,	F	90,
Fis	85 ⅓,	G	80 ⅖,	Gis	76 ²⁄₁₅,	A	71 ⁷⁄₁₀,	B[♭]	67 ⅕,	H	60.

The uncorrected cent amounts by which the 5ths are tempered are

C–G	–8.635,	G–D	0,	D–A	–5.643,	A–E	0,
E–H	–7.228,	H–Fis	0,	Fis–Cis	–10.955,	Cis–Gis	+4.542,
Gis–Dis	+5.314,	Dis–B[♭]	+6.856,	B[♭]–F	–7.711,	F–C	0.

The reader may verify that these amounts add up to –23.460 cents, that is, minus one ditonic comma. Werckmeister presents the list as a monochord-oriented variant of Werckmeister IV. Since C♯–G♯ is pure in any variant of that temperament, the string length for Cis is wrong and must be corrected to 114 ⅕. Let us modify the list accordingly. The corrected list reads (cf. Table 8):

C	120,	Cis	114 ⅕,	D	107 ⅕,	Dis	101 ⅕,	E	95 ⅗,	F	90,
Fis	85 ⅓,	G	80 ⅖,	Gis	76 ²⁄₁₅,	A	71 ⁷⁄₁₀,	B[♭]	67 ⅕,	H	60.

The corrected cent amounts are

C–G	–8.635,	G–D	0,	D–A	–5.643,	A–E	0,
E–H	–7.228,	H–Fis	0,	Fis–Cis	–6.413,	Cis–Gis	0,
Gis–Dis	+5.314,	Dis–B[♭]	+6.856,	B[♭]–F	–7.711,	F–C	0.

The reader may verify that the sum equals –23.460 cents as before. Evidently, we can change the list in any desired way by modifying one string length at a time; the sum will remain unaffected.

Formal Proof.

Let an instrument without subsemitones be tuned in a correct temperament by a tuner who does not know or pretends not to know what a ditonic comma is, but nevertheless tunes all twelve 5ths playable. Each 5th has a string length ratio of the form $(3:2)R$ where R may be called an impurity ratio. For instance, the impurity ratio of C–G in the above example can be calculated as (120 : 80 ⅖)/(3 : 2) = (80 : 80 ⅖). A pure 5th has an impurity ratio of unity.

Consider a trip around the circle. The string length ratio of the large interval obtained by concatenating the twelve 5ths is $(3:2)^{12}Q$ where Q is the product of the twelve impurity ratios. Let the twelve 5ths be followed by or intermixed with seven octaves in the opposite direction; this will, in any correct temperament, cause the roundtrip to end at the starting point. Since $(3:2)^{12}$ equals 531441 : 4096 and seven octaves can be represented by $(2:1)^7 = 128:1$, there holds

$$(531441 : 4096)Q / (128 : 1) = 1 : 1$$

from which $Q = 524288 : 531441$. This ratio is the inverse of one ditonic comma; it translates into –23.460 cents. The twelve impurity ratios similarly translate into the twelve amounts in cents by which the 5ths are tempered (negative cent amount for a flat 5th, positive for a sharp 5th, zero for a pure 5th); we have shown that the sum of these twelve amounts equals minus one ditonic comma regardless of which correct temperament was tuned.

In practice this means that the tuner may concentrate on the temperament's musical qualities and its suitability for the intended repertoire, and keep some mental distance to theory. Was Werckmeister aware of this? He probably was, but the relevant fact is that his presentation of the ditonic comma is hidden away in the theoretical and monochord-oriented parts of the *Musicalische Temperatur*.

4. The Comma Was and Remained Syntonic

In the *Musicalische Temperatur,* the commatis means the 81 : 80 syntonic comma unless otherwise indicated. For tangible evidence, Werckmeister begins his first chapter with a critical comment on meantone temperament as follows: "Some put forward, all 5ths must beat ¼ commatis flat, in which case all 3rds would be and remain quite pure"

(Quote 11.) "Einige bringen vor, es müsten alle quinten ein Viertel commatis herunter schweben, so würden hingegen alle Tertien gantz rein seyn und bleiben" --p.1.

Only the syntonic comma produces pure major 3rds when used as indicated. For another piece of evidence, consider the string length ratios

A–e	3 : 2	quinta;
Adur–e	40 : 27	quinta commata deficiens.

These two lines (--pp.48-49; Quote 12) are entries in what Werckmeister calls a "very useful little Lexicon" of theoretical ratios. By dividing the latter ratio into the former one should get a ratio for the commatis. The result is 81 : 80.

Werckmeister uses his "little Lexicon" (which is more than twelve pages long) to establish that four consequtive pure 5ths calculated from F♯ onwards cause C♯, G♯, D♯, and A♯ to fall within the respective small intervals defined by pairs of certain theoretical tones and subsemitones. He compares the A♯ obtained in this way with a subsemitone called B[♭] molle, defined such that B[♭] molle–f is a pure 5th. The comparison shows that the A♯ is not identical to B[♭] molle; it misses by a quantity described as "a small differens (more about that below) which amounts to about the width of a compass-mark"

(Quote 13.) "... eine kleine differens (davon drunten ein mehres) welche etwa einen Circul-Stich austräget" --p.52.

The tone denoted B[♭] molle is located at string length ratio 405 : 512 relative to F♯, according to the Lexicon. By dividing a 64 : 81 Pythagorean 3rd into that ratio, we find that the differens equals 32805 : 32768, or 1.954 cents. This is the well-known schisma, or the difference between the ditonic and the syntonic comma. What we see is the ditonic comma sneaking in through the back door.

Werckmeister provides more about the differens as promised: The ratio 32805 : 32768 appears explicitely on his p.63. This time it is said to extend “hardly” the thickness of “a small compass-mark on a 3-ft monochord” (“kaum einen kleinen Circulstich auff einem 3füssigen Monochordo” – Quote 14) and to cause some difficulties on the monochord as mentioned in the preceding Section.

The differens shows up again implicitely in the beginning of *Kapitel XXIV* where the relative merits of correct temperament vs. meantone temperament are illustrated with a trip around the circle-of-5ths: If the 5ths are pure, then “the end point will exceed the starting point by a ratio that amounts to a little more than a commatis and causes that tiny interval to be increased. Since it will be forced $^{12}/_{4}$, that is, 3 commata downwards when all 5ths are made to beat $^{1}/_{4}$ commatis flat, it is easily seen that [the end point] falls $^{8}/_{4}$ commatis short of the starting point, and that only $^{4}/_{4}$ should be used to flatten some of the 5ths in a correct temperament, so that the point where we started will be reached again.”

> (Quote 15.) “Denn wenn wir durch alle quinten hindurch gehen, ... bleibet eine proportion, die den terminum, woraus der Anfang der quinten gemachet worden, gar ein klein wenig mehr als ein commatis überschreitet, und selbes subtile intervallum erhöhet. Da ... hingegen durch den Umgang der durch ein Vierthel com. herunter gelassenen quinten 12 Vierthel com. herunter kommen, also ist durch die Vernunfft leicht abzunehmen, daß noch 8 Vierthel com. weiter herunter von dem Anfangs-Punkte geschritten worden, und daß in einer richtigen temperatur nur vier Vierthel in etlichen quinten herunter schweben müssen, wenn der Punct, woraus wir gegangen, wieder erlanget werden soll” --p.64.

(Note the side remark: The effect of the differens — which was considered too small to be engraved separately, cf. Quote 9 on page 11 — is to enlarge the comma. We are still in the monochord-oriented part of the book; while *Kapitel XXIV* does not mention the monochord explicitely, *Kapitel XXIII* and *XXV* both do.) In the above passage Werckmeister first refers to the ditonic comma indirectly but correctly, then omits the differens in two places where modern theorists would have included it; »a little less than $^{8}/_{4}$ com.« and »a little more than $^{4}/_{4}$« would have been blameless. The passage serves to justify the view that meantone temperament was “an entirely spurious and unrhymed thing” (“ein gantz falsch und ungereimt Ding” --p.64; Quote 16). Werckmeister may have felt that this opinion of his was best conveyed without pedantic detail.

Full detail is resumed promptly: The $^{4}/_{4}$ of a comma and the differens go hand in hand in the very next passage. This happens in two side remarks in a text whose main purpose is to remind the reader that the number of tempered 5ths in a correct temperament does not necessarily have to be four: “It is also possible to divide these $^{4}/_{4}$ and the small differens into some other number of parts, as the temperaments can be arranged in various ways. However, the beating must not get flatter than the excessus in the circle of pure 5ths requires it: Here, also the small differens is to be divided [along with] the commata, as the sensus cannot comprehend it”

> (Quote 17.) “Es können auch diese 4 Viertheil und die kleine differens in andere Theile getheilet werden, weil die Temperaturen auf unterschiedliche Weise können angestellet werden. Die Schwebung kan aber nicht weiter herunter kommen, als der excessus in den Zirckel der reinen quinten es erfordert: Allhier wird auch die kleine differens in die commata eingetheilet, weil sie der sensus nicht begreiffen kan” --pp.64-65.

Here again, the differens is treated as a quantity to be added to the comma. The enlarged comma, as well as the expressions “these $^{4}/_{4}$ and the small differens,” “the excessus in the circle of pure 5ths,” and “a little more than a comma” describe the ditonic comma without introducing it as a concept or assigning a name to it. (Incidentally, it was stated on page 6 that the *Musicalische Temperatur* contains evidence in favour of the standard modern interpretation. The last sentence in the above quote — implying that the differens should be added to the [syntonic comma] before the sum is distributed among the tempered 5ths — is one of just two pieces of such evidence in the book. The other is the sizes of the tempered 5ths in the specification table --p.78.)

A full theory of the ditonic comma is presented beginning a few lines later (--pp.65-66). Werckmeister first recapitulates the difference between meantone temperament and a “right” (i.e., correct) temperament: “... if all 5ths in the keyboard were pure, ... the excess would, as said, be a little more than a comma; in [meantone temperament] where the 5ths are ¼ comma flat there is a defect of 2 commata before the radix from which the start was made will be hit; consequently, only some 5ths must be flattened in a right temperament; for the amount by which the perfectly pure 5ths ascend above the octave, that amount and nothing more must again be compensated for, so that the Octava relative to the root where the start was made can become pure.”

> (Quote 18.) “... wenn alle quinten im Clavir reine wären, ... ist, wie gesaget, der Excess ein klein wenig über ein comma; In diesen durch ¼ com. niedergelassenen ist ein defect, welcher 2 commata in sich hält, ehe die radix, aus welcher den Anfang gemachet worden, getroffen wird; Also müssen nur etliche quinten in der rechten Temperatur hernieder gelassen werden, denn wieviel die vollkommenen reinen quinten über die octav gestiegen, so viel und nicht mehr muß wieder ersetzet werden, damit der Octava mit der Wurzel, worinn wir den Anfang gemachet, möge rein bleiben” --p.65.

Here again, the differens goes unmentioned in connection with meantone temperament; the “2 commata” should be understood as something like »a little less than 2 commata«. (The “Octava” obviously refers to a practice of laying the temperament with a bearing plan that ends an octave higher than it started, as prescribed in the step-by-step tuning instructions for Werckmeister IV; the last sentence of the quote suggests that this final octave was listened to for checking purpose.) The quantitative part follows: The number $81^3 = 531441$ is introduced. The number 262144 is derived by keeping track of string lengths during a trip in pure 5ths around the circle. Going an Octava down from the end point does not bring us back to the starting point; indeed, “this number, when doubled, does not return to the root (which was C) from which it originated, but it gives 524288 which, when compared with 531441, is exceeded by that number by an 81 : 80 Comma and an additional small differens 32805 : 32768. This small differens extends, as already mentioned above, roughly the width of a thin line engraved with the compass ...”

> (Quote 19.) “... wenn nun diese letzte Zahl 262144 dupliret wird, so kommt sie nicht wieder zum 531441 als ihrer Wurzel, (welche C gab) woraus sie entsprungen, sondern sie giebet 524288, welche, wenn sie mit der Wurzel 531441 examiniret und überleget wird, so wird diese von jener ein Comma 81 : 80 und noch eine kleine differens 32805 : 32768 ... überschritten. Von dieser kleinen differens ist schon oben erwehnet worden, daß sie in der Operation des Circuls etwa einer subtilen Linien breit austrage ...” --p.66.

A little later, Werckmeister urges his readers to verify the result and, for the first time in the *Musicalische Temperatur*, presents a string length ratio for the ditonic comma: “As a check, one may add the two ratios 81 : 80 and 32805 : 32768; the outcome will again be the ratio 531441 : 524288.”

> (Quote 20.) “Zur Probe addire man diese beyden proportiones 81 : 80 und 32805 : 32768 so wird diese proportion 531441 : 524288 wieder heraus kommen” -- p.66.

To add the ratios is geometrical monochord language; arithmetically, the ratios should of course be multiplied. There follows some analysis of tones, subsemitones, and ratios, after which Werckmeister leaves the subject with the remark, “Incidentally, ... 531441 : 524288 is known to have been called a comma by our predecessors”

> (Quote 21.) “Sonsten ist bekannt, daß ... 531441 : 524288 bey den Alten ein comma genennet worden” -- p.68.

To read “die Alten” as “the ancients” with allusion to Pythagoras may or may not be correct; the expression appears in Werckmeister’s *Erweiterte und verbesserte Orgel-Probe* in connection with Michael Praetorius (1571-1621) and his writings.

The above quotes and summaries cover essentially all there is about the ditonic comma in the *Musicalische Temperatur*. By including everything, the author hopes to minimise the risk of selection bias in the interpretation that follows below.

Werckmeister’s presentation of the ditonic comma is complete and mathematically correct. The three omissions of the differens in connection with meantone temperament can be explained as intentional simplification for the sake of clarity. (Epistemologically, a complete and perfect theory is not needed in order to argue that meantone is not a good temperament; it is sufficient to point out one or more shortcomings.) However, the ditonic comma is hidden away in the book’s theoretical chapters where it is introduced piece by piece, to a large extent indirectly and in side remarks. It is not referred to in the part of the book (essentially *Kapitel XXX*) where the practical tuning guides are found. The string length ratio 81 : 80 for the syntonic comma appears again and again in the *Musicalische Temperatur*; a practical method for constructing it with the compass is sketched on pp.63-64; an approximate method for dividing it into two, three, or four equally large parts is discussed on p.37. No such method is given for the 531441 : 524288 ditonic comma, and its string length ratio appears only twice in the book: On p.66 as the outcome of the proposed check, and on p.68 where “die Alten” are said to have called it a comma. The latter statement appears as a side remark to a side remark at the end of the book’s *Kapitel XXV*, after which Werckmeister leaves the subject for good.

Werckmeister’s remark that the ratio 531441 : 524288 had been called a comma by “die Alten” implies that he considered it to be of historical interest. It might also imply that his readers should feel free to call it a comma if they wanted to; but it does not imply that Werckmeister wanted them to use it for tuning his temperaments.

The ratio 531441 : 524288 is systematically downplayed into a subordinate role in the *Musicalische Temperatur*. It is not given a name of its own, Werckmeister does not himself call it a comma, and it is nowhere described as being useful for practical tuning purpose. Werckmeister does not say that 531441 : 524288 is the quantity that tuners should distribute among some of the 5ths. Instead, he introduces the differens as a small but theoretically unavoidable quantity and says that the 81 : 80 comma with the differens added to it is the quantity that monochord users should distribute. Thus, the 81 : 80 syntonic comma plays the principal part. (Note that the syntonic comma is used in the book’s *Kapitel XXI* to discuss the 5ths in Werckmeister III, as observed in Section 3 on page 11).

The conclusion that can be extracted from the *Musicalische Temperatur* is the following:

Experienced meantone tuners who wanted to tune Werckmeister’s temperaments did not need to be trained in how to use the ditonic comma. This applies to those tuners who were used to think of the syntonic comma when tuning meantone temperament, as well as those who did not pay much of a thought to any type of comma during the practical tuning process.

5. Subjective Tuning and the Good Triads

It will be argued in this Section that Werckmeister expected his readers to use the same subjective tuning techniques for tuning his temperaments as they would have used for tuning meantone temperament. An academic analysis of old tuning practice will not be attempted; it would become very long, and only a few aspects are relevant for our purpose. These aspects can be summarised (and oversimplified) as follows.

- Tempered intervals were tuned melodically. A skilled tuner could judge the size of a tempered interval accurately by listening to the tones one of a time. Quantitative methods such as beat counting were not used.
- Tempered intervals were not always tuned to their theoretically correct sizes. Meantone temperament with nominally equal-sized 5ths was sometimes tuned with slightly unequal-sized 5ths in practice, with audible key colouring as a result. The major 3rds were sometimes slightly sharp in cases where they should theoretically be pure.

Various meantone temperaments in which the 5ths were flat by amounts in the vicinity of ⅕ comma were formulated theoretically. Sharp major 3rds are explicit in these temperaments, some of which are thought to have been used more or less often. More detailed information can be found in Jorgensen's *Tuning* (much of the author's knowledge of the subject originates from that book) and in Lindley's *Temperaments* article (2001a) in the *New Grove*.

It is useful to consider the tuning instructions written by Gottfried Keller, the German composer and harpsichordist who settled in England at some time before 1694, at which time he is known to have obtained a visa to visit the Continent. Keller tuned root-position triads directly: In each triad he tuned first the major 3rd, then the 5th; each triad was completed before going to the next, according to Jorgensen who bases his Almost ⅕ Ditonic Comma Meantone Temperament (*Tuning*, pp.55-61) on

> "Keller's words,
> Observe all ye Sharp thirds must be as Sharp as ye eare will permit, And all fifths as flat as the eare will permit" (*Tuning*, p.55; Quote 22).

A corresponding passage can be found in Werckmeister's step-by-step instructions for Werckmeister IV. It is longer, because the instructions assume 5th-wise tuning: "One lets the first 5th, namely G against C, beat a little bit; G and d are pure, and so is the [downwards] octave d–D; A must again be flat from D; A and e must again be pure: Now one holds the first key C against e, and it will be found that the e beats very slightly sharp against C, which one can hardly discern with the ear. Since, however, the slight beatings in C–G and D–A will not always be hit accurately enough with the ear as to make C–e tolerably tempered, one must correct until it sounds tolerably, which can be done easily, because one knows where [the error] is to be found."

> "So lasset man die erste quinta, nemlich G gegen C ein klein wenig schweben, G und d sind rein, ingleichen die Octava d–D; das A muß gegen das D wieder herunterwarts schweben; A und e müssen wieder rein seyn; Nun halte man den ersten clavem C gegen e, so wird sich befinden, daß das e gegen C ein gar weniges, welches man fast mit dem Gehör nicht penetriren kan, herauffwarts schwebet: Weil aber die kleinen Schwebungen in C–G und D–A durch das Gehör nicht allemal so accurat getroffen wird, daß das Temperament C–e erträgligh ist, muß man so lang corrigieren, bis sie erträglich klingen, welches gar leicht geschehen kan, weil man weiß, wo es zu finden ist ..." -- p.79. (Quote 23)

Note the two pieces of evidence for the use of subjective tuning methods when tuning correct temperaments: Werckmeister wants us to go back and correct until C–e sounds "erträglich", not until it beats at some specified rate; and Werckmeister's "klein Wenig" is similar to Keller's meantone-oriented "as Sharp as ye eare will permit" in the sense that tuners must be skilled in the use of subjective techniques in order to tune on the basis of them. Since "ein klein Wenig" appears in the

Musicalische Temperatur without explanation or guidance, Werckmeister must have assumed that those readers who knew how to give and to take would obtain the intended results when they relied on experience they had acquired by tuning meantone temperament.

The *Musicalische Temperatur* contains step-by-step tuning instructions only for Werckmeister IV. The instructions can readily be adapted to Werckmeister III and V, as the underlying principle is the same; this may explain why Werckmeister did not write instructions out in full for each temperament. (In the monochord-oriented discussion of the 5ths in Werckmeister III — *Kapitel XXI*, --pp.57-58 — the tones are treated in the same order and downwards octaves are included in the same places as in the tuning instructions for Werckmeister IV. However, it provides little information about practical tuning, as the amount by which G beats flat against C is simply given as "¼ comm." without being restated in subjective terms.)

Everything considered, it appears reasonable to assume that the above conclusion — no retraining needed for a meantone tuner who wanted to tune Werckmeister IV — holds for Werckmeister III and V as well. Whether or not it also holds for the "Septenario" Werckmeister VI temperaments is without implications for the present purpose.

Werckmeister's remark that one can hardly discern the beats in C–e with the ear is a mystery, since it seems to imply that they are more difficult to hear than those in C–G. Perhaps Werckmeister wanted those of his readers who had not previously listened to beats in a decim to be aware of the possibility that they might have to listen a couple of times before hearing them. Could it be that C–e should beat more slowly than C–G? Such a hypothesis does not appear tenable. Although variants of Werckmeister IV with that property can be formulated, some 5ths have to be flattened by substantially more than ⅓ comma; this causes such variants to deviate even more from Werckmeister's string-length defined version than the standard modern interpretation does (the latter being defined as a strict ⅓ ditonic comma version). It is worth observing here that C–E beats faster than C–G in the string-length defined version of Werckmeister IV. The two intervals are equal-beating in Variants III-*a* and IV-*a* to be derived later in this report, because these Variants were so designed.

The quote from Keller's tuning instructions implies, according to Jorgensen, that the major 3rds in Keller's meantone temperament were sharp by the amount by which the 5ths were flat. The triads C–E–G and D–F–A in Werckmeister III and IV have that property too, according to Werckmeister's specification tables. Some tuners in the 17th century may have found it natural to try and get these triads to sound with a quality not too inferior to meantone temperament. (Triads are mentioned several times in the *Musicalische Temperatur* and there can be no doubt that Werckmeister considered the quality with which they sounded to be significant: He proposes on the last page ofhis *Vorrede* that monochords should ideally be equipped with three strings so that one can "present Triades, Syzigias and everything to the ear;" and he states on his p.59 that B–D♯ and other major 3rds in Werckmeister III sound tolerably, especially when a third tone is added to form a full "Trias or Syzigia Harmoniaca.")

A hypothetical tuning experiment

In an attempt to be specific as to what is meant by tuning Werckmeister III from a meantone viewpoint, let us conclude this Section with a scenario that could be either old or modern.

> A meantone tuner has just finished tuning an instrument. The temperament can be described as the tuner's personal version of some meantone temperament belonging to the ⅕ comma family. The tuner has checked the good triads and verified that they sound with the best harmonious quality the tuner can achieve. Now, the tuner decides to retune the instrument into Werckmeister III.

> The tuner's first thought is to leave C, E, and G unchanged, because that triad sounds as desired already. If one now retunes A and F so that A–E and F–c become pure, as they must be in Werckmeister III, then the major 3rd F–A gets the same size as C–E. Therefore, if one retunes D so that D–A in one's personal variant of Werckmeister III gets the size that A–E had before retuning started, then the triad D–F–A is very likely to sound as desired; for it will sound exactly as A–C–E did in the tuner's personal version of meantone temperament except for being transposed. The remaining six tones may be obtained by tuning six pure intervals: E–B and F–B♭–D♯–G♯–C♯–F♯.

The above illustrates, on the basis of subjective arguments alone, the elementary fact that if the tuner has decided how C–E–G and D–F–A should sound, then the tuner's personal variant of Werckmeister III is fully determined and there are no more decisions to be made. (Note that the order in which the tones were retuned in this experiment is not supported in the *Musicalische Temperatur*.)

> Now, suppose our tuner realises that the 5th G–D gets too flat if the above idea is followed slavishly. Indeed, since Werckmeister III is a ¼ comma temperament, in order for C–E and C–G to be sharp and flat respectively by the same amount, that amount must be larger than it would be in ⅕ comma meantone temperament. Thus E and G will need to be modified, assuming that C is kept at standard pitch; also, D–A must be flattened more than suggested above. The tones A and B must, of course, be modified along with E such that A–E and E–B remain pure; and D must be retuned so that G–D and D–A get suitably tempered. Presumably, our tuner would want to carry out these refinements in such a way that the triads C–E–G and D–F–A suffer the least possible loss of harmonic quality.

If a method could someday be found by which the harmonic quality of a triad could be expressed quantitatively as a function of the sizes of its tempered intervals, then the »least possible loss« would become a theoretically well-defined concept that could be handled with standard mathematical methods. Cent values could then be calculated to approximate in a well-defined manner a tuner's personal version of Werckmeister III as tuned with meantone-oriented subjective techniques.

Our present knowledge of old tuning techniques is insufficient for such an experiment to produce useful results, and the idea will not be pursued further. A more earth-bound approach will be outlined in the next Section.

6. Methodology

We shall now assume that some tuners in the 17th century would understand the specification table for Werckmeister III as follows, and tune the temperament accordingly.

Recipe for Tuning Werckmeister III

(*i*) The 5ths C–G, G–D, D–A, and B–F# are flat. All other 5ths are pure.
(*ii*) In the major triad C–E–G, the major 3rd is as sharp as the 5th is flat.
(*iii*) In the minor triad D–F–A, the major 3rd is as sharp as the 5th is flat.
(*iv*) The 5th B–F# should not be tempered more strongly than G–D.

The individual parts are supported by evidence in the *Musicalische Temperatur*; the way they are combined here is not. Recipe (*iv*) has been added because any modern interpretation of Werckmeister III should probably be harmonically balanced in the strict sense described by Jorgensen. (Very briefly, harmonic balance means that modulation into a dominant or subdominant key with one more accidental does not result in a purer sound. The major 3rds in all of Werckmeister's specification tables fulfil this condition; those derived from his string length list for Werckmeister IV show minor deviations from it).

Once the tuner has choosen a pitch standard and the size of one of the tempered intervals, the Recipe defines the rest of the temperament. It would appear natural to tune C to pitch standard, then tune G such that C–G sounds as desired. When E is tuned several steps later (or in the very next step if triads are tuned directly), the tuner might want to arrange for the major 3rd C–E to be as sharp as the 5th is flat. The modern theorist will need to ask a question at this stage:

Quantitative Definition Wanted.

What does it mean for a major 3rd and a 5th to be tempered by the same amount?

What we need is one or more methods by which the modern theorist can calculate cent values for the temperament in a well-defined way. We shall work with three methods — not because they have musical merit, but because they exist.

Definition (*a*): Equal 5:4 and 3:2 beat rates;
Definition (*b*): Equal 5:4 and 6:4 beat rates;
Definition (*c*): Same amount in cents.

Cent values based on each definition will be calculated in the following Sections. Here, it will be pointed out — in order to prevent misunderstandings — that none of the above Definitions are historically justified.

Triads containing equal-beating intervals exist in certain meantone temperaments of to the ⅕ comma family. Hovewer, this is unlikely to have been a design consideration at the time when these temperaments were originally devised, and we can conclude only that the existence of equal-beating intervals did not cause the tuners to reject the temperaments in which they appeared. Lord Stanhope's remark from 1806, "And, from the equality of the beatings, equal deviations from perfection is ascertained" (Jorgensen, p.284; Quote 24) refers to consequtive 5ths along the circle, not to a major 3rd compared with a 5th.

Many, including the author, have gotten the impression that equal-beating intervals sometimes cause certain chords to sound harmonious when the tones are allowed to sound simultaneously. It appears that the phenomenon has never been thoroughly investigated. Could it be an illusion? Methods to determine if a subjective impression is real exist, but tend to be costly. If some director of a pharmaceutical research company would undertake a full-scale double-blind study as a post-retirement project free of charge and, for example, present the result in units of harmoniousness as a function of the difference between the relevant beat rates, that might provide us with valuable insight. (Note that beat rates are never exactly equal on non-electronic instruments.) Jorgensen argues that equal beat rates reduce the chaos among upper harmonics (*Tuning*, p.383), but also calls them "merely another theory on paper" (*ibid.*, p.44). Equal-beating intervals will be used in this report as merely another way of computing cent values.

Definition (*b*) is based on the fact that a 3rd may be tempered more than a 5th and still sound tolerable. In *Kapitel XV* of the *Musicalische Temperatur*, after having stated that the perfect consonants are the octaves, the 4ths and the 5ths, Werckmeister urges his readers to "try if not a 4th sounds as dissonant as a 5th when a comma is given to it or taken from it; in contrast, a major 3rd may endure much more, and is more tolerable to hear when a comma is removed or added..."

> (Quote 25.) ...so versuche man, ob eine Quarta nicht eben so falsch klinget als eine Quinta, wenn man derselben ein Comma gibet, oder nimmet, hingegen leidet eine Tertia major vielmehr, und ist erträglicher zu hören, wenn ein Comma ab- oder zugethan wird... (--p.30).

Lateron, Werckmeister goes so far as to classify the "admittedly somewhat harder" 3rds C♯–F, G♯–c, and F♯–B♭ in Werckmeister IV, which are ⅔ of a comma sharp, as quite pleasant "in vollem concent" as mentioned in Section 2 (Quote 5 on page 9).

A subjective equivalent of Definition (*b*) may be formulated as follows,

> Definition(*bb*): The 5th and the major 3rd sound equally impure.

Quote 25 above seems to imply that, in order for a 5th and a major 3rd to sound equally impure, the major 3rd must be tempered more than the 5th. — How much more? A factor of two looks like a useful guess in the absence of quantitative historical evidence. As a first attempt, one might try and tune the major 3rd sharp by twice as many cents as used to flatten the 5th. The author tried, but found that the temperaments obtained did not look reasonable on paper. If instead we make the (5:4) major 3rd to beat twice as fast as the (3:2) 5th, results look better. The author elected to reformulate »twice as fast« into »same beat rate at the 5:4 and 6:4 levels« in Definition (*b*), because it makes sense to compare the beats at a level where the tones in the minor triad D–F–A have a common higher harmonic. This reformulation is purely cosmetical in the absence of inharmonicity. (The reformulation is not used in the presentation of results in Tables 2 and 6; all beat rates of 5ths in all Tables refer to the 3:2 level, and inharmonicity is ignored throughout).

Definition (*c*) is cent-based. Logarithms had been available from around 1630 and were used in computing-intensive areas such as astronomy and navigation; but string lengths continued to be used for music theory, perhaps because of their monochord-friendliness and their ability to express intervals as pairs of whole numbers. Werckmeister informs us (--p.37) that the ¼ comma marks on the *Kupferblatt* were constructed "only mechanicé" (that is, by dividing the comma into four equal distances with the compass), and observes that the method is slightly inaccurate because it ignores the comma's variation in size along the string. As to the reason why an approximation has to be used, Werckmeister points out that a proportio superparticularis such as 81 : 80 is not the square of any ratio of whole numbers. Werckmeister observes that his approximation divides the comma according to the string lengths 324 : 323 : 322 : 321 : 320. On his *Kupferblatt*, the comma thus divided is shown to the right of the engraved E and to the left of the engraved G, suggesting that monochord users who had studied the *Kupferblatt* may have tended to sharpen C–E by 324 : 323 and flatten C–G by 321 : 320. The two ratios differ in size by about 0.050 cents. This is a small amount — small enough, in the author's opinion, for Definition (*c*) to qualify as a modern approximation to its string-length based historical counterpart.

7. Variant III–a: Derivation

We shall now compute frequencies and cent values for Werckmeister III in accordance with Definition (*a*) and Recipe (*i*)-(*iv*) of the preceding Section. This will cause C–E–G and D–F–A to become equal-beating triads. The common beat rate of F–A and D–A is not required to be identical to the common beat rate of C–E and C–G. It is convenient to adopt a pitch standard. It will be chosen arbitrarily as A = 440 Hz. Conversion into any other desired pitch standard is straightforward; one simply scales all frequencies and beat rates by the appropriate factor.

The derivation will proceed 5th-and-4th-wise within the octave from middle C upwards. Thus, Werckmeister's 5ths-up-octaves-down bearing plan will not be used in the derivation. Beat rates will sometimes refer to 4ths; such intervals should be understood as inverted 5ths and will be denoted as 5ths in the text and in the Tables. There should be no risk of misunderstanding, as the frequency of each tone will be stated. The formulation of a historically justifiable bearing plan for tuning the temperament on a given instrument will be left to the reader; the task is rich in subtleties and falls outside the scope of this report.

The tuner must choose the precise size of the 5th C–G. Let us assume that the tuner lets it beat flat at the rate of $x = 2\tfrac{1}{2}$ Hz. It is practical to calculate the various frequencies in non-standard order as follows.

$$A = 440 \text{ Hz};$$
$$E = 3A/4 = 330 \text{ Hz (because A–E is pure)}.$$

Since x is also the beat rate of C–E and since F–C is pure, we find

$$C = (4E - x)/5 = (3A - x)/5 = 263\tfrac{1}{2} \text{ Hz};$$
$$G = (3C - x)/2 = (9A - 8x)/10 = 394 \text{ Hz};$$
$$F = 4C/3 = (12A - 4x)/15 = 351\tfrac{1}{3} \text{ Hz}.$$

The beat rate of the sharp major 3rd F–A is

$$4A - 5F = 4A - (12A - 4x)/3 = 4x/3 = 3\tfrac{1}{3} \text{ Hz}.$$

We want this to be the 3:2 beat rate of the flat 5th D–A as well, in virtue of Recipe (*iii*) and Definition(*a*). Thus the third harmonic of D must be $4x/3$ higher than the second harmonic of A,

$$3D = 2A + 4x/3 = (6A + 4x)/3;$$

one-third of that gives us the frequency of the fundamental,

$$D = (6A + 4x)/9 = 294\tfrac{4}{9} \text{ Hz}.$$

The remaining tones are obtained easily because the intervals are pure:

$$B = 3E/2 = 9A/8 = 495 \text{ Hz};$$
$$B\flat = 4F/3 = (48A - 16x)/45 = 468\tfrac{4}{9} \text{ Hz};$$
$$E\flat = 2B\flat/3 = (96A - 32x)/135 = 312\tfrac{8}{27} \text{ Hz};$$
$$A\flat = 4E\flat/3 = (384A - 128x)/405 = 416\tfrac{32}{81} \text{ Hz};$$
$$C\sharp = 2A\flat/3 = (768A - 256x)/1215 = 277\tfrac{145}{243} \text{ Hz};$$
$$F\sharp = 4C\sharp/3 = (3072A - 1024x)/3645 = 370\tfrac{94}{729} \text{ Hz}.$$

Frequencies with decimals are listed in Table 1 together with cent values relative to equal temperament. The latter were rounded to three decimal places in a manner that ensures strictly equal-sized pure 5ths and zero accumulated rounding error around the circle. The amounts by which the major 3rds are sharp are listed too; they show, among other things, that Variant III-*a* is harmonically balanced. For which other values of x would that be the case? In order to obtain an upper limit, it is sufficient to observe that B–F♯ must not be tempered more than G–D measured in cents, cf. Recipe (*iv*) on page 20. The 4:3 beat rates are

$$3G - 4D = (3A - 376x)/90,$$
$$3B - 4F\sharp = (111A + 32768x)/29160.$$

The latter interval is located one major 3rd higher than the former on the keyboard, so it is allowed to beat up to a factor $5/4$ faster:

$$(111A + 32768x)/29160 \leq (5/4) \cdot (3A - 376x)/90.$$

This reduces to

$$x \leq 138A/23131 = 2.62505 \text{ Hz};$$

thus one could choose C–G to beat as fast as 2.625 Hz and still obtain a harmonically balanced temperament. The limit is slightly conservative because an approximation has been used: B–F♯ is located a sharp major 3rd higher than G–D, not a pure major 3rd as the ratio $5/4$ assumes.

A lower limit for x has to be based on subjective judgment, as no strict criterion has been found. Two of the above formulas show that G–D increases in size while B–F♯ decreases when x is decreased, but there is probably nothing gained by letting them differ very much in size. Let us postulate that B–F♯ should be tempered by at least two-thirds as many cents as G–D. That means, in terms of slightly conservative beat rates,

$$(111A + 32768x)/29160 \geq (2/3) \cdot (5/4) \cdot (3A - 376x)/90,$$

which reduces to

$$x \geq 699A/134288 = 2.29 \text{ Hz}.$$

As a conclusion, any beat rate in the range $2.29 \text{ Hz} \leq x \leq 2.625 \text{ Hz}$ could be worth trying. The rate of $x = 2\frac{1}{2}$ Hz was chosen as a round number near the upper end of that interval. It makes C–G flat by 5.484 cents, or a little more than one-quarter of a syntonic comma. Note that the choice of an appriopriate size for C–G is the only point in the derivation where the comma is alluded to.

Readers who use a C tuning fork might prefer to derive the frequencies in a different order: G, E, and F from C; A from E; D from A and F; the remaining tones as above.

8. Variant III–b: Derivation

We shall use Definition (b) and Recipe (i)-(iv) of Section 6 (page 20). Let y be the common beat rate of the (5:4) major 3rd C–E and the (6:4) 5th C–G. An example of a useful beat rate is $y = 3\,{}^{7}\!/_{8}$ Hz. The derivation proceeds essentially as in the foregoing Section:

$$
\begin{aligned}
A &= 440 \text{ Hz;} \\
E &= 3A/4 = 330 \text{ Hz;} \\
B &= 9A/8 = 495 \text{ Hz;} \\
C &= (4E - y)/5 = (3A - y)/5 = 263\,{}^{9}\!/_{40} \text{ Hz;} \\
G &= (6C - y)/4 = (18A - 11y)/20 = 393\,{}^{139}\!/_{160} \text{ Hz;} \\
F &= 4C/3 = (12A - 4y)/15 = 350\,{}^{29}\!/_{30} \text{ Hz.}
\end{aligned}
$$

The major 3rd F–A beats at the rate $4A - 5F = 4A - (12A - 4y)/3 = 4y/3$. We want the (6:4) 5th D–A to beat at that rate too; this implies $6D - 4A = 4y/3$, or

$$D = (6A + 2y)/9 = 294\,{}^{7}\!/_{36} \text{ Hz.}$$

Proceeding with pure 4ths and 5ths in the flatwise direction from F, we get

$$
\begin{aligned}
B\flat &= 4F/3 = (48A - 16y)/45 = 467\,{}^{43}\!/_{45} \text{ Hz;} \\
E\flat &= 2B\flat/3 = (96A - 32y)/135 = 311\,{}^{131}\!/_{135} \text{ Hz;} \\
A\flat &= 4E\flat/3 = (384A - 128y)/405 = 415\,{}^{389}\!/_{405} \text{ Hz;} \\
C\sharp &= 2A\flat/3 = (768A - 256y)/1215 = 277\,{}^{373}\!/_{1215} \text{ Hz;} \\
F\sharp &= 4C\sharp/3 = (3072A - 1024y)/3645 = 369\,{}^{2707}\!/_{3645} \text{ Hz.}
\end{aligned}
$$

The (4:3) beat rates of G–D and B–F♯ are

$$
\begin{aligned}
3G - 4D &= (6A - 457y)/180, \\
3B - 4F\sharp &= (111A + 32768y)/29160;
\end{aligned}
$$

the range of potentially useful beat rates y can be estimated in the same way as in the preceding Section:

$$({}^{2}\!/_{3}) \cdot ({}^{5}\!/_{4}) \cdot (6A - 457y)/180 \;\leq\; (111A + 32768y)/29160 \;\leq\; ({}^{5}\!/_{4}) \cdot (6A - 457y)/180;$$

this reduces to

$$699A/94463 \;\leq\; y \;\leq\; 2208A/250621,$$

or, since $A = 440$ Hz,

$$3.256 \text{ Hz} \;\leq\; y \;\leq\; 3.87645 \text{ Hz.}$$

The beat rate $y = 3\,{}^{7}\!/_{8}$ Hz was chosen as a round number close to the upper end if this interval. Frequencies with decimals, cent values, and selected beat rates are shown in Table 2 on page 37.

9. Variant III–c: Derivation

Consider the two major 3rds C–E and F–A. Both are sharp by the same amount, because the 5ths F–C and A–E are pure in Werckmeister III. Let us call the amount x. It will be expressed in cents, because we are going to use Definition (c) of Section 6.

Recipes (*ii*) and (*iii*) of Section 6 (page 20) imply that both of the two 5ths C–G and D–A are flat by the amount x. Let G–D be flat by some amount y, and let B–F♯ be flat by the amount $y - w$ where the size of w will be left open for a moment; Recipe (*iv*) implies $w \geq 0$. We shall write down two equations from which x and y may be found. First, the sum of the amounts by which the tempered 5ths are flat must equal one ditonic comma P; thus, $x + y + x + (y - w) = P$. Secondly, let S be the syntonic comma. Since C–E is sharp by one syntonic comma minus the sum of the amounts by which C–G, G–D, D–A, and A–E are flat, we have $S - x - y - x - 0 = x$. (As on page 10, the zero simply expresses the fact that A–E is pure.) Our two equations may be written

$$2x + 2y = P + w;$$
$$3x + y = S.$$

Solving for x and y gives

$$x = \tfrac{1}{2}S - \tfrac{1}{4}P - \tfrac{1}{4}w,$$
$$y = \tfrac{3}{4}P - \tfrac{1}{2}S + \tfrac{3}{4}w;$$

furthermore,

$$y - w = \tfrac{3}{4}P - \tfrac{1}{2}S - \tfrac{1}{4}w.$$

It may be easily shown from these formulas that any value of w greater than zero would cause G–D to be flatter than any other 5th in the temperament. However, there appears to be no musical advantage in tuning G–D flatter than B–F♯. The value $w = 0$ was therefore adopted for Variant III-c. Cent values, frequencies derived from the cent values, and selected beat rates derived from the frequencies are shown in Table 3 on page 38. If one accepts the notion that the amounts $\tfrac{1}{12}P$ and $\tfrac{1}{11}S$ are good approximations for one another, one may formulate Variant III-c in a way that looks elegant on paper:

C–G and D–A are $\tfrac{5}{24}$ ditonic comma flat;
G–D and B–F♯ are $\tfrac{7}{24}$ ditonic comma flat.

All other 5ths are, of course, pure. (It is unlikely that Werckmeister was aware of the approximation mentioned above, and it is far from certain that he would have used it even if he was. It is not a mathematical theorem, but merely a numerical near-miss that might have been hard to reconcile with the unity-and-perfection-based philosophy to which several chapters in the *Musicalische Temperatur* are devoted. The approximation is used here because it leads to a formulation of the temperament that is easy to remember.)

10. Other Correct Temperaments

Werckmeister IV is defined in the second table on p.78 in the *Musicalische Temperatur*. It has seven tempered 5ths: C–G, D–A, E–B, F♯–C♯, and B♭–F are ⅓ commatis flat; A♭–E♭ and E♭–B♭ are ⅓ commatis sharp. All major 3rds are sharp: ⅓ commatis for C–E, F–A, B♭–D, G–B, D–F♯, A–C♯, and E–G♯; ⅔ commatis for E♭–G; 3⁄3 commatis for B–D♯; and 4⁄3 commatis for F♯–B♭, C♯–F, and A♭–C. Is it possible to interprete these specifications in the way we did for Werckmeister III in Section 6? An attempt to do so may, for instance, proceed as follows.

"Recipe" for Tuning Werckmeister IV

(*I*) The 5ths C–G, D–A, E–B, F♯–C♯, and B♭–F are flat; A♭–E♭ and E♭–B♭ are sharp; all other 5ths are pure.

(*ii*) In the major triad C–E–G, the major 3rd is as sharp as the 5th is flat.

(*iii*) In the minor triad D–F–A, the major 3rd is as sharp as the 5th is flat.

Note that "Recipe"(*ii*) and (*iii*) are identical to the entries with the same numbers in the Recipe of Section 6 (page 20). This means that Definitions (*a*) through (*c*) can be applied to the two involved triads. There is no "Recipe" (*iv*); the condition that B–F# should not be flattened more than G–D is trivially fulfilled in Werckmeister IV because both 5ths are pure. However, harmonic balance is not fulfilled automatically but must be imposed.

Variant IV-a: Derivation.

Variant IV-*a* can be derived from "Recipe" (*ii*)-(*iii*) and Definition (*a*) of Section 6 (page 20) as follows. Let x be the common beat rate of the 3:2 flat 5th C–G and the 5:4 sharp major 3rd C–E. We find

$$E = \tfrac{3}{4} A,$$

$$C = \tfrac{3}{5} A - \tfrac{1}{5} x,$$

$$F = \tfrac{4}{3} C = \tfrac{4}{5} A - \tfrac{4}{15} x \quad \text{(because F–C is pure)},$$

$$G = \tfrac{9}{10} A - \tfrac{4}{5} x;$$

and, since G–D is pure in Werckmeister IV,

$$D = \tfrac{3}{4} G = \tfrac{27}{40} A - \tfrac{3}{5} x.$$

As in Section 7, the beat rate of F–A is $\tfrac{4}{3} x$ because F–C and A–E are pure. The 3:2 beat rate of D–A is

$$3D - 2A = \tfrac{81}{40} A - \tfrac{9}{5} x - 2A = \tfrac{1}{40} A - \tfrac{9}{5} x.$$

We want the beat rates for F–A and D–A to be equal, in accordance with "Recipe" (*iii*) and Definition (*a*):

$$\tfrac{4}{3} x = \tfrac{1}{40} A - \tfrac{9}{5} x.$$

Solving for x yields $x = \tfrac{3}{376} A$. For $A = 440$ Hz, this means a common beat rate of $x = 3.51$ Hz. Substituting this into some of the above formulas gives $C = \tfrac{225}{376} A$, $G = \tfrac{42}{47} A$, $D = \tfrac{63}{94} A$, and $F = \tfrac{75}{94} A$. The 5th C–G gets a string length ratio of 112 : 75; it is 7.7115 cents flat, or a little less than ⅓ ditonic comma. This agrees well with the corresponding entry in the specification table for Werckmeister IV, suggesting that the methodology cannot be entirely wrong.

But how do we proceed from here? The methodology, when applied to Werckmeister IV, does not tell us how to get the rest of the tones. In order to get the temperament finished in some way or another, let us postulate that we want a harmonically balanced variant with several proportional-beating triads. If we arrange for the major 3rd in E–G–B to beat twice as fast as the triad's 5th, and do the same for F♯–A–C♯ and B♭–D–F, we obtain: $B = {}^{843}\!/_{752}\,A$; $F\sharp = {}^{2529}\!/_{3008}\,A$; $C\sharp = {}^{15107}\!/_{24064}\,A$; $B\flat = {}^{552}\!/_{517}\,A$; $A\flat = {}^{45321}\!/_{48128}\,A$. It remains to find a suitable value for $E\flat$. If we let the 4th E♭–A♭ and the 5th E♭–B♭ beat at the same rate (which, like the other artificially imposed conditions, is without historical merit), we find

$$E\flat = (3A\flat + 2B\flat)/7 = {}^{2626089}\!/_{3705856}\,A.$$

Werckmeister did not like ratios with large numerators or denominators. The above ratios are used here because they are exact; frequencies calculated from them are accurate to any number of decimals. Frequencies and cent values for Variant IV-*a* are listed in Table 5 on page 40.

Variant IV-b: Derivation.

Similarly, Definition(*b*) on page 20 may be used with "Recipe" (*ii*)-(*iii*) above to derive Variant IV-*b*. Let y be the common beat rate of the 5:4 sharp major 3rd C–E and the 6:4 flat 5th C–G:

$$4E - 5C = y = 6C - 4G.$$

This implies, since $E = {}^{3}\!/_{4}\,A$,

$$C = (3A - y)/5,$$

$$G = (18A - 11y)/20.$$

Since F–C and G–D are pure, we have

$$F = {}^{4}\!/_{3}\,C = (12A - 4y)/15,$$

$$D = {}^{3}\!/_{4}\,G = (54A - 33y)/80.$$

The beat rate of F–A is ${}^{4}\!/_{3}\,y$, because F–C and A–E are pure. We want it to equal the 6:4 beat rate of D–A, in accordance with "Recipe" (*iii*) and Definition (*b*):

$${}^{4}\!/_{3}\,y = 6D - 4A = (2A - 99y)/40.$$

Solving for y yields

$$y = {}^{6}\!/_{457}\,A = 5.78 \text{ Hz for } A = 440 \text{ Hz}.$$

Substituting this into the above results gives

$$C = {}^{273}\!/_{457}\,A;\quad G = {}^{408}\!/_{457}\,A;\quad D = {}^{306}\!/_{457}\,A;\quad F = {}^{364}\!/_{457}\,A.$$

To introduce additional proportional-beating triads turns out not to work well with Definition (*b*). A simple approach that appears to work is to flatten E–B, F♯–C♯, and B♭–F as much as possible consistent with harmonic balance; this means string length ratio 408 : 273 for E–B (i.e., same ratio as C–G) and 457 : 306 for F♯–C♯ and B♭–F (same ratio as D–A). This gives

$$B = {}^{102}\!/_{91}\,A;\quad F\sharp = {}^{153}\!/_{182}\,A;\quad C\sharp = {}^{457}\!/_{728}\,A;\quad A\flat = {}^{1371}\!/_{1456}\,A;\quad B\flat = {}^{222768}\!/_{208849}\,A.$$

It could make sense to choose $E\flat$ such that the two sharp 5ths become equal-sized. In that case,

$$E\flat = ({}^{1}\!/_{2} \cdot A\flat \cdot B\flat)^{1/2} = ({}^{459}\!/_{914})^{1/2}A = 311.807 \text{ Hz for } A = 440 \text{ Hz}$$

where the factor ½ appears because one of the sharp 5ths has been inverted into a 4th. Frequencies for Variant IV-*b* as obtained in this way are listed together with other details in Table 6 on page 41.

Variant IV-c: Derivation.

To derive Variant IV-c from Definition (c) on page 20 is easy and problem-free. Let z denote the amount in cents by which C–G and D–A are flat and C–E is sharp; let S be the syntonic comma. Since G–D and A–E are pure in Werckmeister IV, we have $z = S - z - 0 - z - 0$, from which $z = \frac{1}{3} S$. We may, for instance, use that amount to temper not only C–G and D–A but also the other flat 5ths, and let the sharp 5ths A♭–E♭ and E♭–B♭ share the schisma; this makes the sharp 5ths a little purer than specified. The cent values obtained in this way are shown in Table 7 on page 42 together with frequencies derived from them.

The version of Werckmeister IV defined by the table of string lengths on p.80 in the *Musicalische Temperatur* is shown in Table 8. A »standard modern interpretation« of Werckmeister IV in which all tempered 5ths are exactly ⅓ ditonic comma flat or sharp is shown in Table 9.

Werckmeister V (No Variant Derived).

Werckmeister V is specified in the table on p.79 in the *Musicalische Temperatur*. There are six tempered 5ths: F–C, D–A, A–E, F♯–C♯, and C♯–G♯ are ¼ commatis flat; G♯–D♯ is ¼ commatis sharp. All major 3rds are sharp: 2⁄4 commatis for F–A, C–E, G–B, D–F♯, A–C♯, and E–G♯; ¾ commatis for B–D♯, F♯–B♭, B♭–D, and E♭–G; and 4⁄4 commatis for C♯–F and A♭–C. The Recipe of Section 6 can hardly be adapted to this temperament, since C–G is pure and all major 3rds are at least twice as sharp as the flat 5ths are flat. Still, the basic idea — to assign priority to specifications related to one or more frequently used triads in which all intervals are tempered — can be used for Werckmeister V. An interpretation based on proportional-beating triads F–A–C and D–F–A was constructed, but was rejected because it was found not to look convincing on paper. An interpretation in which all flat 5ths are tempered by ¼ syntonic comma and the sharp 5th is one schisma purer than specified (in analogy with Variant IV-c) was constructed and found to look more reasonable. Variants with six equal-sized major 3rds in which the flat 5ths alternate between two sizes (as they do in Variant IV-b) can be constructed and might be worth studying. Cent values will not be presented here and Werckmeister V will not be considered further.

11. Summary of Results

The methodology outlined in the foregoing Sections performs very differently for the three temperaments considered.

- It works without problems for Werckmeister III.
- It can be applied to Werckmeister IV, but it does not define all the tones.
- It is not useful for Werckmeister V.

In other words, the methodology works for the temperament for which it was devised but is not generally applicable. The specification tables for Werckmeister IV and V contain seven and six good major 3rds respectively; these are specified to be equal-sized, suggesting that these temperaments were designed to resemble meantone temperament in that respect. However, strictly equal-sized major 3rds may not have been an absolute requirement, as slightly unequal-sized good major 3rds appear in the string-length defined version of Werckmeister IV. Variants IV-*b* and IV-*c* were derived in this report with seven strictly equal-sized major 3rds for the lack of better ideas. Some major 3rds in Variant IV-*a* are almost as impure as the least playable major 3rds in the standard modern interpretation of Werckmeister IV (compare C♯–F in Table 5 with the same interval in Table 9); they are less impure in Variants IV-*b* and IV-*c* (Tables 6 and 7).

In the case of Werckmeister III, the picture is similar: Variant III-*a* resembles the standard modern approximation in that the key colour contrast between C major and G major is more abrupt than specified; the contrast is reduced in Variants III-*b* and III-*c* to slightly less than the specified ¼ syntonic comma, in accordance with the hypothesis that meantone tuners in the 17th century might have found the contrast unusually large and wanted to reduce it a little. (It may be shown that VariantIII-*c* is the only possible interpretation of Werckmeister III in which both of the major 3rds G–B and D–F♯ are precisely twice as sharp as C–E, measured in cents; Variant III-*c* can, in fact, be derived from that condition alone.) Perhaps the arguments become easier to follow if the sharpness of C–E and G–B (i.e., the amount by which the intervals are larger than a pure major 3rd) is listed in table form:

(Table) Werckmeister III: Major 3rds C-E and G-B Compared

WERCKMEISTER III: MAJOR 3RDS C–E and G–B COMPARED

	SMI†	III-*a*	III-*b*	III-*c*	Specs
Sharpness of C–E (cent)	3.911	3.282	5.090	4.888	+¼‡
Sharpness of G–B (cent)	9.776	8.766	9.343	9.776	+2/4
(Sharpness of G–B) / (Sharpness of C–E)	2.50	2.67	1.84	2.00	2,00
(Sharpness of G–B) – (Sharpness of C–E)	5.865	5.484	4.253	4.888	¼

†SMI: Standard Modern Interpretation. ‡One-quarter of a syntonic comma equals 5.376 cents.

It was suggested in Section 2 (on page 8) that the sharpness ratio of 2.50 between G–B and C–E for the standard modern interpretation is too large given that the ratio is specified as 2. But then, the above table indicates that Variant III-*a* with its ratio of 2.67 is no better. If instead of a ratio we consider the difference, then its 5.484 cents, despite still exceeding ¼ syntonic comma, might be taken as a marginal improvement over the standard modern interpretation in which the difference is 5.865 cents. For Variant III-*c*, the ratio of 2.00 matches specifications while the difference of 4.888 is slightly below specifications. In Variant III-*b*, the ratio of 1.84 is smaller than specified, and so is the difference of 4.253 cents; thus, Variant III-*b* agrees with the hypothesis that tuners in the 17th century might have wanted to reduce the colour contrast between C–E and G–B a little.

A tendency for Definitions (*b*) and (*c*) (page 20) to produce more reasonable results than Definition (*a*) can be discerned in the above observations. The tendency persists if we proceed to Werckmeister IV and compare the string-length defined version of that temperament with the Variants derived for it. Before doing so, let us look briefly at a few pecularities in Werckmeister's presentation of his string length list:

The step-by-step tuning instructions for Werckmeister IV appear at the end of Werckmeister's *Kapitel XXX* together with some remarks about certain major 3rds being "somewhat harder" ("etwas härter" --p.80; Quote 26) followed by comments on the minor 3rds. Much of *Kapitel XXXI* appears to have been written with the purpose of refuting what Werckmeister considered to be polemic responses to his earlier writings. The very first paragraph, however, contains the string-length list. Werckmeister begins his *Kapitel XXXI* as follows: "This temperament will also be presented briefly and slightly differently on the monochord: If the string is divided into 120 equal parts, then C falls at the 120th point, C♯ at 114 [$\frac{1}{5}$], D 107 $\frac{1}{5}$, D♯ 101 $\frac{1}{5}$, E 95 $\frac{3}{5}$, F 90, F♯ 85 $\frac{1}{3}$, G 80 $\frac{2}{5}$, G♯ 76 $\frac{2}{15}$, A 71 $\frac{7}{10}$, B[♭] 67 $\frac{1}{5}$, H 64, c 60."

> "Diese Temperatur wird auch auff ein weniges und kleine differens ohne weitläufftigen Proceß auff dem Monochordo vorgestellet: Wenn die Saite in 120 gleiche Theile getheilet wird, so fallet das C auff den 120sten Punct[,] Cis auff 114 $\frac{1}{2}$, D 107 $\frac{1}{5}$, Dis 101 $\frac{1}{5}$, E 95 $\frac{3}{5}$, F 90, Fis 85 $\frac{1}{3}$, G 80 $\frac{2}{5}$, Gis 76 $\frac{2}{15}$, A 71 $\frac{7}{10}$, B[♭] 67 $\frac{1}{5}$, H 64, c 60" --p.80. (Quote 27)

(The erroneous string length for C♯ has been corrected in the English text but not in the German.) Werckmeister continues: "If thereupon a string is drawn and two bridges are placed, one at the 120th and the other one division before the first division, and furthermore a movable bridge that can be easily shifted back and forth, then a tolerable temperament will be obtained when an instrument or a clavichordium is tuned from it. However, it must be treated carefully, for if the bridge is not kept per pendicular when moved from one point to another, or the string is bent, then it will not work. Also, a string tends to get out of tune when the bridge is moved back and forth; one must listen regularly and check if the string, when left alone, is still pure against the C from which one started. Carelessness and hastework may result in misconception and should be avoided." (--pp.80-81; Quote 28)

In the rest of the *Kapitel*, which is written as a single two-pages-long paragraph, Werckmeister reflects in a seemingly depressed mood about "many kinds of opinion" ("vielerley Meinungen" --p.81; Quote 29) that he appears to have received from certain readers of the *Orgel-Probe*. He classifies some of these "Meinungen" as "Injurien" (--p.82; Quote 30), but does not specify beyond the argument that his temperaments, which he himself considered comprehensible even to beginners, had been misunderstood.

The fact that Werckmeister placed his string length list next to complaints about polemics appears, at least to the author, to carry information of its own. In the Foreword, Werckmeister told us that a "wohlgeübter Musicus practicus" who knew how to give and to take did not need to be taught how to tune an instrument (cf. Quote 3 on page 7); but in *Kapitel XXXI*, certain readers apparently need to be told how to use a monochord. Perhaps the string length list and the accompanying beginner's guide to the monochord were included in order to discourage polemically inclined readers from using an old trick: Tune intentionally wrong, then complain that the temperament does not work. An alternative hypothesis could be: The list was proposed by someone else as part of some debate, and Werckmeister included the list in order to close the debate.

In any case, the string length list was published in the *Musicalische Temperatur*. It should be safe to assume that it represents a valid way of tuning Werckmeister IV. We do not need to assume that it is the *only* valid way.

In the preceding Section, we applied a methodology designed for Werckmeister III to the specification table for Werckmeister IV. Did the methodology reproduce the string-length defined version of Werckmeister IV? The idea that it should is somewhat wild, and in the real world we can only hope to get close. Comparison makes sense only for those tones that the methodology actually determines. There are only four of them, as A is fixed to the pitch standard and A–E is pure. If we compare the cent values obtained for the Variants with those derived from Werckmeister's string length list for these tones, the following happens:

(Table) Werckmeister IV: Four Tones Compared

WERCKMEISTER IV: FOUR TONES COMPARED

	F (cent)	C (cent)	G (cent)	D (cent)	RMS† (cent)
Standard Modern Interpretation	7.820	9.775	3.910	5.865	1.82
Variant IV-*a*	9.076	11.031	5.274	7.229	3.11
Variant IV-*b*	6.093	8.048	3.650	5.605	1.38
Variant IV-*c*	6.518	8.473	3.259	5.214	1.08
Werckmeister's String length list	6.458	8.413	1.733	3.688	—

†RMS: Root-Mean-Square = square root of the average of the difference squared between the cent value for the Variant and the cent value as derived from the string length list (Table 8), the average being taken over the four tones.

It is seen from this comparison that, for all four tones, Variants IV-*b* and IV-*c* are closer to the string-length list defined version than the standard modern interpretation is. Hovever, they still miss the target by more than one cent. Variant IV-*a* disappoints, as it moves all four tones in the opposite direction. This is the second time that Definitions (*b*) and (*c*) produce more credible results than Definition (*a*).

In view of the above, and after having inspected the sizes of the major 3rds in the various Variants of Werckmeister III as they appear on paper, the author tends to have a little more confidence in Variants III-*b* and III-*c* than in Variant III-*a* .

12. Concluding Remarks

From time to time, complaints are heard that Werckmeister III has its weaknesses. Here are two recent examples:

> Modern Objection No. 1: "Bumpy"
>
> "Of [well-tempered] systems, Werckmeister III is notable for its purity in the best keys and its suitability for organs with large quint mixtures (many of the fourths and fifths are in tune); but it is irregular and bumpy in the way it deals with modulation and key colour" (Stephen Bicknell, 1997). (Quote 31)

> Modern Objection No. 2: "Melodic Bumpiness"
>
> "Werckmeister III and 'Kellner' share a serious problem in their treatment of melody... Melodic leaps up and down to A♭ and D♭ can come across like singing with poor breath support as the tuning of these notes is so unexpectedly low. But, so is the note A in the simple melody F–A–C ! ...These latter two temperaments have plenty of faithful and enthusiastic fans, especially due to the way they sound reasonably good in *earlier* music (based mostly on regular mean-tone layouts). But it cannot be denied that their melodic bumpiness borders on the effect of randomness, by the 18th-century standard itself" (Bradley Lehman, 2005-II). (Quote 32)

Bumpiness in Werckmeister III can in part be ascribed to the manner in which Werckmeister's specification table is customorarily being interpreted. The standard modern interpretation is based on one single viewpoint: Rigid usage of the ditonic comma. This makes all major 3rds one schisma smaller than specified, with the exception of those that are Pythagorean in size, and leads to more key colour contrast between certain frequently used keys than there should be according to the specification table, because C–E and F–A become disproportionately pure in comparison with G–B and B♭–D as observed in Section 2 on page 8. Variants III-*b* and III-*c* as derived in Sections 8 and 9 with emphasis on the quality of the good triads relative to meantone temperament attempt to mitigate the contrasts a little by keeping C–E and F–A closer to specifications. The author believes that these Variants are meaningful approximations to Werckmeister III as it sounded in the 17th century. Modulation in a ¼ comma well temperament can, of course, never be as smooth as in a ⅕ or ⅙ comma well temperament.

A singer used to be accompanied by an instrument tuned in ⅙ comma meantone temperament might indeed find the A in the standard modern interpretation of Werckmeister III to be unexpectedly low relative to F and C; Variants III-*b* and III-*c* should mitigate this, too.

The best way to evaluate the Variants is, of course, to tune and play them. By doing so, the author has so far gotten the impression that Variants III-*b* and III-*c* present the music better than Variant III-*a*. One difficulty lies in the choice of repertoire. To play Bach's keyboard music in Werckmeister III is out of question now that the quest for Bach's tuning system is making progress (see the Postscript below); music composed for meantone temperament can as well be played in meantone temperament — and the repertoire written specifically for Werckmeister III appears to be limited.

As to Werckmeister's attitude to the ditonic comma, consider again his side remark "...and causes that tiny interval to be increased" ("...und selbes subtile intervallum erhöhet" — part of Quote 15 on page 14). The remark contains no independent information and does not qualify as a third piece of evidence in favour of the standard modern interpretation of Werckmeister III (cf. Page 15); it merely anticipates the result of Werckmeister's derivation of the ratio 531441 : 524288 and, like other allusions to the ditonic comma in the *Musicalische Temperatur*, is hidden away in a side remark in the book's theoretical or monochord-oriented part without being referred to in the *Kapitel* that contains the practical tuning instructions. Nevertheless, the remark lends support to the following formulation:

> The commatis generally means the syntonic comma — but it gets larger if one obtains it from a trip around the circle-of-5ths.

Werckmeister says in one of his later publications (not covered in this narrowly focused technical report) that he might in practice tune the three consequtive tempered 5ths in Werckmeister III as in ¼ comma meantone temperament. Doing so in strict accordance with theory would cause D–F♯ to become purer than G–B in disagreement with harmonic balance as implied by the specification table, but this can be remedied with a little fine adjustment. It is quite possible that Werckmeister and/or some of his contemporaries tuned a harmonically balanced version with some resemblance to Variant III-*c* or even Variant III-*b*; but direct historical evidence is not at hand.

Postscript regarding Bach's temperament

POSTSCRIPT

The ornament on top of of the title page of the WTC is now thought by some if not all researchers to contain information about a temperament that Bach considered useful for playing those pieces and possibly useful for other keyboard music (Lehman 2005 I-II; note that earlier work by others was credited in the supplementary material on the Internet).

Assuming that this is true, to what extent did Bach follow Werckmeister's ideas? Modern authors sometimes touch upon this question in a manner as if they were not fully aware of a few facts:

- Unequal-sized flat 5ths exist in Werckmeister VI *a*-*b*. They exist also in Werckmeister IV as defined by the string length list on p.80 in the *Musicalische Temperatur* (cf. Table 8).
- Tempered 5ths in the chromatic part of the circle appear in all temperaments described in the *Musicalische Temperatur*, with Werckmeister III as the only exception.
- Sharp 5ths appear in all temperaments described in the *Musicalische Temperatur*, with Werckmeister III as the only exception.

As for sharp 5ths, Werckmeister's summary of the content of his *Kupferblatt* contains an amusing passage: "Num. III is a correct temperament, which is divided evenly through ¼ comma, and where some 5ths are pure, whereas some beat ¼ comma sharp, and some beat flat."

> (Quote 33.) "Num. III ist eine richtige Temperatur, welche ebenmässig durch ¼ comma eingetheilet wird, da etliche quinten rein, etliche aber ¼ comma aufwerts, etliche aber unterwärts schweben" (--p.56).

Werckmeister seems to have forgotten for a moment that Werckmeister III does not contain sharp 5ths. This detail adds support to the view that one or two sharp 5ths in a correct temperament was regarded by Werckmeister as being normal.

Bach was not the kind of person who needed a mentor. Werckmeister, on his part, regarded his readers as fully competent, and he did not want to act as a mentor for them. This can be seen from a passage in his Foreword: Just as his intention in one of his previous publications was "not to prescribe anything to any outstanding Musico, as I find myself much too humble for that, and would commit a huge mistake" — similarly, in the *Musicalische Temperatur* "no experienced Musico will be prescribed how he should tune a tempered keyboard instrument,"

> (Quote 34.) "Gleichwie ich in meinen Musicalischen Wegweiser keinen vornehmen Musico etwas vorzuschreiben gemeinet, sintemal ich mich viel zu gering dazu befinde, und eine grosse Schwachheit begehen würde: Also wird auch in diesem Tractat keinen erfahrenen Musico, wie er ein clavier temperiert stimmen solle, aufgebürdet" (--second last page of the *Vorrede*).

At this point in the Foreword there follows the passage quoted in Section 1 (Quote 3 on page 7), after which Werckmeister continues: "In this book, I demonstrate to those who are eager to learn it, how one can formulate and arrange the temperaments in various ways. One may place the beatings of the 5ths in whichever keys one wants; it is just that the perfect consonants should not be treated too much. It is enough when a keyboard is so tempered that it is usable throughout"

> (Quote 35.) "Ich bezeuge hierinnen der Lehrbegierigen, wie man die Temperaturen einrichten, und auf unterschiedliche Arten anstellen könne, es mag einer die Schwebung der quinten hinbringen in welche claves er will, nur daß den perfecten consonantien nicht zu viel gethan werde, genug ist es, wenn ein clavier so temperiret wird, daß es durchaus wohl can gebraucht werden" (--two last pages of the *Vorrede*).

Here, the perfect consonants are the 5ths and to treat them means to temper them. In the above passage, Werckmeister grants his readers the freedom to design and use correct temperaments of their own, the only restriction being playability in all keys. A closer look reveals that one more restriction is imposed in the book's *Kapitel XXVIII*: "...Even though the temperament can be made in many ways, all major 3rds must beat sharp and the minor 3rds flat" ("...ob man schon auf vielerley Weise die Temperatur haben kan, so müssen doch die Tertiæ majores herauf, die minores herunter schweben..." --p.75; Quote 36); this applies unconditionally to all correct temperaments, as is clear from remarks in the very beginning of the *Kapitel*, the title of the *Kapitel*, the last ten lines of the previous *Kapitel*, and elsewhere in the book.

Harmonic balance is not mentioned in Quote 35 above. A requirement of harmonic balance may possibly have been implicit as part of the concept of usability ("wohl kan gebraucht werden"), but not in the strict sense defined by Jorgensen, as two minor local valleys exist in the distribution of the major 3rds in the string-length defined version of Werckmeister IV (p.80 in the *Musicalische Temperatur*; Table 8 in this report). Lehman's proposed Bach temperament has a similar local valley at F♯–B♭, which is unusual but not against Werckmeister's principles.

The sizes of the major 3rds in Lehman's proposed Bach temperament peak at E–G♯, which is considered by modern theorists to be unusual. (Lehman refers in an ongoing debate to temperaments by Neidhardt, who was considered by Bach to be a greater tuning theorist than Werckmeister, cf. Lindley 2001b. Note that Bach must have been familiar with Werckmeister's tuning principles because of geographical and social circumstances.) In Werckmeister's temperaments, the major 3rds do not reach their maximum size until F♯–B♭ or C♯–F. In his presentation of Num. 3 in the *Musicalische Temperatur*, Werckmeister remarks that "there is a comma too much between C♯ and F and [this] is one of the hardest major 3rds; but it cannot be otherwise, for since it is quite seldomly used, it is better to sweep the hardness to there than into those that are used more often" (Vom cis ins F befindet sich ein comma zu viel, und ist eine von den härtesten tertien mit, es kan auch nicht anders seyn, denn weil dieselbe gar selten gebrauchet wird, ist is besser, dass man die Härtigkeit dahin schantzet, als in dieselben, so zum öfftern gebraucht werden --p.58; Quote 37). There are similar but briefer remarks about F♯–B♭ and A♭–C. Less categorical words are used in these remarks than in the requirement of sharp major 3rds; furthermore, the remarks appear in connection with Werckmeister III and do not, at least not explicitely, refer to correct temperaments in general. Perhaps one may say that a distribution of major 3rds that peaks at E–G♯ disagrees with all of the temperaments that Werckmeister proposed but does not disagree with the overall principles outlined in the *Musicalische Temperatur*.

Comments to Lehman's interpretation have addressed the presence of tempered 5ths among the chromatic keys, unequal-sized flat 5ths, and (in particular) a sharp 5th, on grounds that none of these features exist in Werckmeister III (or Vallotti, or Kellner, or Barnes). Objections of this kind appear to originate from a mistaken belief that Werckmeister III is representative for Werckmeister's temperaments. Werckmeister III may be the temperament that Werckmeister himself advocated, a view that seems to originate from Christiaan Huygens, Sorge and Marpurg (Lindley, 2001b); but it is atypical, and only part of Werckmeister's ideas can be inferred from it. Both Lehman's interpretation of Bach's ornament, and those other interpretations that the author has seen so far, are — as should be clear from the above — in full agreement with the principles of correct temperament as presented by Werckmeister in the *Musicalische Temperatur*. People have, in the author's opinion, to some extent been discussing non-existent problems. Bach constructed a correct temperament of his own — one that suited his needs better than any of those proposed by Werckmeister. By doing so, Bach did exactly what Werckmeister expected his readers to do. To look for differences in opinion between Bach and Werckmeister is unlikely to produce new insight. It is, in fact, as useless as nodum in scirpo quære.

13. Tables

Frequencies, cent values, and beat rates are presented on the following pages. They were calculated for the hypothetical case of an instrument without inharmonicity. Strings on real keyboard instruments are typically made from material whose elastic stiffness affect the frequencies of the various harmonics in a way that causes frequencies of the higher harmonics to deviate slightly from the values they would have for a perfectly flexible string. This in turn causes the beat rates to differ from the values in the tables. The beat rates of 5ths and 4ths are generally not very much affected, and one may often tune on the basis of them with good results. The effect upon the beat rates of 3rds is larger. In fact, on a modern upright piano, inharmonicity may cause some major 3rds to beat more than ten percent faster than they should according to the tables; one should be aware of this when listening to beats in major 3rds for checking purpose, and when tuning 3rds directly.

TABLE 1

Variant III-a

Frequencies, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.50000	+12.359	F#	370.12894	+0.629
C#	277.59671	+2.584	G	394.00000	+8.831
D	294.44444	+4.590	Ab	416.39506	+4.539
Eb	312.29630	+6.494	A	440,00000	0.000
E	330.00000	+1.955	Bb	468.44444	+8.449
F	351.33333	+10.404	B	495.00000	+3.910

Tempered 5ths:

	Tempering (cent)	Beat rate† (Hz)	
C – G	-5.484	2,50	cf. C – E
G – D	-6.195	4.22	
D – A	-6.545	3.33	cf. F – A
B – F#	-5.236	4.48	

†Cents refer to 5ths. Beat rates refer to (4:3) 4ths or (3:2) 5ths regardless of text.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+3.282	2.50	cf. C – G
G – B	+8.766	10.00	
D – F#	+9.725	8.29	
A – C#	+16.271		
E – Ab	+16.271		
B – Eb	+16.271		
F# – Bb	+21.506		
C# – F	+21.506		
Ab – C	+21.506		
Eb – G	+16.022		
Bb – D	+9.827		
F – A	+3.282	3.33	cf. D – A
C – E	+3.282	2.50	

TABLE 2

Variant III-b

Frequencies of tones, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.22500	+10.552	F#	369.74266	-1.178
C#	277.30700	+0.777	G	393.86875	+8.254
D	294.19444	+3.120	Ab	415.96049	+2.732
Eb	311.97037	+4.687	A	440,00000	0.000
E	330.00000	+1.955	Bb	467.95556	+6.642
F	350.96667	+8.597	B	495.00000	+3.910

Tempered 5ths:

	Tempering (cent)	Beat rate† (Hz)	
C – G	-4.253	1.9375	cf. C – E
G – D	-7.089	4.83	
D – A	-5.075	2.58	cf. F – A
B – F#	-7.043	6.03	

†Cents refer to 5ths. Beat rates refer to (4:3) 4ths or (3:2) 5ths regardless of text.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+5.090	3.875	cf. C – G
G – B	+9.343	10.66	
D – F#	+9.388	8.00	
A – C#	+14.463		
E – Ab	+14.463		
B – Eb	+14.463		
F# – Bb	+21.506		
C# – F	+21.506		
Ab – C	+21.506		
Eb – G	+17.253		
Bb – D	+10.164		
F – A	+5.090	5.17	cf. D – A
C – E	+5.090	3.88	

TABLE 3

Variant III-c

Frequencies and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.25562	+10.753	F#	369.78568	-0.977
C#	277.33926	+0.978	G	393.77009	+7.820
D	294.16271	+2.933	Ab	416.00889	+2.933
Eb	312.00667	+4.888	A	440.00000	0.000
E	330.00000	+1.955	Bb	468.01000	+6.843
F	351.00750	+8.798	B	495.00000	+3.910

Tempered 5ths:

	Tempering (cent)	Beat rate† (Hz)	
C – G	-4.888	2.23	cf. C – E
G – D	-6.842	4.66	
D – A	-4.888	2.49	cf. F – A
B – F#	-6.842	5.86	

† Beat rates refer to 4ths or 5ths. Cents refer to 5ths.
The above cent amounts are close to -5/24 P and -7/24 P where P = 23.460 cents is the ditonic comma.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+4.888	3.72	cf. C – G
G – B	+9.776	11.15	
D – F#	+9.776	8.33	
A – C#	+14.664		
E – Ab	+14.664		
B – Eb	+14.664		
F# – Bb	+21.506		
C# – F	+21.506		
Ab – C	+21.506		
Eb – G	+16.618		
Bb – D	+9.776		
F – A	+4.888	4.96	cf. D – A
C – E	+4.888	3.72	

Note that 9.776 and 14.664 cents are very close to multiples of 5/24 ditonic comma.

TABLE 4

Werckmeister III: Standard Modern Interpretation

Frequencies, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.40423	+11.730	F#	369.99442	0.000
C#	277.49582	+1.955	G	393.77009	+7.820
D	294.32876	+3.910	Ab	416.24373	+3.910
Eb	312.18279	+5.865	A	440.00000	0.000
E	330.00000	+1.955	Bb	468.27419	+7.820
F	351.20564	+9.775	B	495.00000	+3.910

Tempered 5ths:

	Tempering (cent)	Beat rate† (Hz)	
C – G	-5.865	2.67	cf. C – E
G – D	-5.865	4.00	
D – A	-5.865	2.99	cf. F – A
B – F#	-5.865	5.02	

† Beat rates refer to 4ths or 5ths. Cents refer to 5ths.
The above cent amounts are -1/4 P
where P = 23.460 cents is the ditonic comma.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+3.911	2.98	cf. C – G
G – B	+9.776	11.15	
D – F#	+9.776	8.33	
A – C#	+15.641		
E – Ab	+15.641		
B – Eb	+15.641		
F# – Bb	+21.506		
C# – F	+21.506		
Ab – C	+21.506		
Eb – G	+15.641		
Bb – D	+9.776		
F – A	+3.911	3.97	cf. D – A
C – E	+3.911	2.98	

TABLE 5

Variant IV-a

Frequencies of tones, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.29787	+11.031	F#	369.93351	-0.285
C#	276.22507	-5.991	G	393.19149	+5.274
D	294.89362	+7.229	Ab	414.33760	-4.036
Eb	311.79818	+3.731	A	440,00000	0.000
E	330.00000	+1.955	Bb	469.78723	+13.405
F	351.06383	+9.076	B	493.24468	-2.240

Tempered 5ths:

	Tempering (cent)	Beat rate (Hz)	
C – G	-7.712	3.51	cf. C – E
D – A	-9.184	4.68	cf. F – A
E – B	-6.150	3.51	cf. G – B
F# – C#	-7.661	4.90	cf. A – C#
Ab – Eb	+5.812	-4.18	(flat 4th)
Eb – Bb	+7.719	-4.18	(sharp 5th)
Bb – F	-6.284	5.11	cf. Bb – D

Cents refer to 5ths. Beat rates refer to 4ths or 5ths.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+4.610	3.51	cf. C – G
G – B	+6.172	7.02	cf. E – B
D – F#	+6.172	5.27	
A – C#	+7.695	4.90	cf. F# – C#
E – Ab	+7.695	7.35	
B – Eb	+19.657		
F# – Bb	+27.376		
C# – F	+28.753		
Ab – C	+28.753		
Eb – G	+15.230		
Bb – D	+7.511	5.11	cf. Bb – F
F – A	+4.610	4.68	cf. D – A
C – E	+4.610	3.51	

Cent amounts refer to 3rds. Two of the beat rates involve tones ouside the octave from middle C upwards.

TABLE 6

Variant IV-b

Frequencies of tones, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	262.84464	+8.048	F#	369.89011	-0.488
C#	276.20879	-6.093	G	392.82276	+3.650
D	294.61707	+5.605	Ab	414.31319	-4.138
Eb	311.80704	+3.780	A	440,00000	0.000
E	330.00000	+1.955	Bb	469.32434	+11.698
F	350.45952	+6.093	B	493.18681	-2.443

Tempered 5ths:

	Tempering (cent)	Beat rate (Hz)	
C – G	-6.353	2.89	cf. C – E
D – A	-7.560	3.85	cf. F – A
E – B	-6.353	3.63	cf. G – B
F# – C#	-7.560	4.84	cf. A – C#
Ab – Eb	+5.963	-4.29	(flat 4th)
Eb – Bb	+5.963	-3.23	(sharp 5th)
Bb – F	-7.560	6.13	cf. Bb – D

Cents refer to 5ths. Beat rates refer to 4ths or 5ths.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+7.593	5.78	cf. C – G
G – B	+7.593	8.63	cf. E – B
D – F#	+7.593	6.48	
A – C#	+7.593	4.84	cf. F# – C#
E – Ab	+7.593	7.25	
B – Eb	+19.909		
F# – Bb	+25.873		
C# – F	+25.873		
Ab – C	+25.873		
Eb – G	+13.556		
Bb – D	+7.593	5.16	cf. Bb – F
F – A	+7.593	7.70	cf. D – A
C – E	+7.593	5.78	

Cent amounts refer to 3rds. Two of the beat rates involve tones ouside the octave from middle C upwards.

TABLE 7

Variant IV-c

Frequencies, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	262.90908	+8.473	F#	369.71589	-1.304
C#	276.14109	-6.518	G	392.73400	+3.259
D	294.55050	+5.214	Ab	414.21164	-4.563
Eb	311.77181	+3.584	A	440.00000	0.000
E	330.00000	+1.955	Bb	469.33333	+11.731
F	350.54544	+6.518	B	492.95452	-3.259

Tempered 5ths:

	Tempering (cent)	Beat rate (Hz)	
C – G	-7.169	3.26	cf. C – E
D – A	-7.169	3.65	cf. F – A
E – B	-7.169	4.09	cf. G – B
F# – C#	-7.169	4.58	cf. A – C#
Ab – Eb	+6.192	-4.45	(flat 4th)
Eb – Bb	+6.192	-3.35	(sharp 5th)
Bb – F	-7.169	5.82	cf. Bb – D

Cents refer to 5ths. Beat rates refer to 4ths or 5ths. The flat 5ths are tempered by -1/3 S where S = 21.50629 cents is the syntonic comma.

Major 3rds:	Tempering (cent)	Beat rate (Hz)	
C – E	+7.169	5.45	cf. C – G
G – B	+7.169	8.15	cf. E – B
D – F#	+7.169	6.11	cf. D – A
A – C#	+7.169	4.56	cf. F# – C#
E – Ab	+7.169	6.85	cf. E – B
B – Eb	+20.529		
F# – Bb	+26.721		
C# – F	+26.721		
Ab – C	+26.721		
Eb – G	+13.361		
Bb – D	+7.169	4.87	cf. Bb – F
F – A	+7.169	7.27	cf. D – A
C – E	+7.169	5.45	cf. C – G

Cent amounts refer to 3rds. Two of the beat rates involve tones ouside the octave from middle C upwards.

TABLE 8

Werckmeister IV: String-Length Defined Version

String lengths, frequencies, and cent values relative to equal temperament:

	String Length	Frequency (Hz)	Tempering (cent)
C	120	262.90000	+8.413
Cis	114 1/5 †	276.25219	-5.821
D	107 1/5	294.29104	+3.688
Dis	101 1/5	311.73913	+3.403
E	95 3/5	330.00000	+1.955
F	90	350.53333	+6.458
Fis	85 1/3	369.70313	-1.364
G	80 2/5	392.38806	+1.733
Gis	76 2/15	414.37828	-3.866
A	71 7/10	440,00000	0.000
B[b]	67 1/5	469.46429	+12.214
H	64	492.93750	-3.319

† Werckmeister's value of 114 ½ for Cis must be a typographical error, as C#–G# is pure.

Tempered 5ths:	Ratio	Tempering (cent)	Beat rate (Hz)
C – G	100 : 67	-8.635	+3.92
D – A	1072 : 717	-5.643	+2.87
E – B	239 : 160	-7.229	+4.125
F# – C#	2560 : 1713	-6.413	+4.10
Ab – Eb	1142 : 759	+5.314	-3.82
Eb – Bb	253 : 168	+6.856	-3.71
Bb – F	112 : 75	-7.712	+6.26

Ratios and cent amounts refer to 5ths. Beat rates refer to 4ths or 5ths.

Major 3rds:	Ratio	Tempering (cent)	Beat rate (Hz)
C – E	300 : 239	+7.229	5.50
G – B	201 : 160	+8.635	9.81
D – F#	201 : 160	+8.635	7.36
A – C#	717 : 571	+7.865	5.01
E – Ab	717 : 571	+7.865	7.51
B – Eb	320 : 253	+20.408	
F# – Bb	80 : 63	+27.264	
C# – F	571 : 450	+25.965	
Ab – C	571 : 450	+25.965	
Eb – G	253 : 201	+12.017	
Bb – D	84 : 67	+5.160	3.50
F – A	300 : 239	+7.229	7.33
C – E	300 : 239	+7.229	5.50

Ratios, cent amounts and beat rates refer to 3rds. Two of the beat rates refer to tones below middle C.

TABLE 9

Werckmeister IV: Standard Modern Interpretation

Frequencies of tones, and cent values relative to equal temperament:

	(Hz)	(cent)		(Hz)	(cent)
C	263.10695	+9.775	F#	369.57684	-1.955
C#	275.93342	-7.820	G	392.88176	+3.910
D	294.66132	+5.865	Ab	413.90013	-5.865
Eb	311.83046	+3.910	A	440.00000	0.000
E	330.00000	+1.955	Bb	469.86328	+13.685
F	350.80927	+7.820	B	492.76912	-3.910

Tempered 5ths:

	Tempering (cent)	Beat rate (Hz)	
C – G	-7.820	3.56	
D – A	-7.820	3.98	
E – B	-7.820	4.46	
F# – C#	-7.820	5.00	
Ab – Eb	+7.820	-5.62	(flat 4th)
Eb – Bb	+7.820	-4.24	(sharp 5th)
Bb – F	-7.820	6.35	

Cents refer to 5ths. Beat rates refer to 4ths or 5ths.
The above cent amounts are plus and minus 1/3 P
where P = 23.460 cents is the ditonic comma.

Major 3rds:	Tempering (cent)	Beat rate (Hz)
C – E	+5.866	4.47
G – B	+5.866	6.67
D – F#	+5.866	5.00
A – C#	+5.866	3.73
E – Ab	+5.866	5.60
B – Eb	+21.506	
F# – Bb	+29.326	
C# – F	+29.326	
Ab – C	+29.326	
Eb – G	+13.686	
Bb – D	+5.866	3.99
F – A	+5.866	5.95
C – E	+5.866	4.47

Cent amounts refer to 3rds. Two of the listed beat rates involve tones ouside the octave from middle C upwards.

REFERENCES

Andreas Werckmeister, “Die Musicalische Temperatur,” Quedlinburg 1691. Facsimile (1996): Guido Bimberg and Rüdiger Pfeiffer (Ed.), Verlag Die Blaue Eule, Annastrasse 74, D-45130 Essen, Germany; <www.die-blaue-eule.de> (accessed 20 May 2000); ISBN 3-89206-736-8.

Owen Jorgensen, “Tuning. Containing: The Perfection of Eighteenth-Century Temperament, The Lost Art of Nineteenth-Century Temperament, and The Science of Equal Temperament, Complete with Instructions for Aural and Electronic Tuning.” Michigan State University Press, East Lansing, 1991. ISBN 0-87013-290-3

Bradley Lehman, “Bach's extraordinary temperament: our Rosetta Stone - 1,” Early Music Vol. XXXIII, No. 1, February 2005, pp. 3-23. Link to Supplementary Material at <www.larips.com>.

Bradley Lehman, “Bach's extraordinary temperament: our Rosetta Stone - 2,” Early Music Vol. XXXIII, No. 2, May 2005, pp. 211-231.

Mark Lindley (2001a), “Temperaments.” *The New Grove Dictionary of Music and Musicians*, edited by Stanley Sadie and John Tyrrell (London: Macmillan 2001), Vol. 25, pp. 248-268.

Mark Lindley (2001b), “Well-tempered clavier,” *ibid.*, Vol. 27, pp. 274-277; see also both articles on *Grove Music Online*, Ed. L. Macy (Accessed 24 Nov 2006), <http://www.grovemusic.com>.

Johan Norrback: “A Passable and Good Temperament. A New Methodology for Stuying Tuning and Temperament in Organ Music.” Göteborg University, Department of Musicology, Box 200, 40530 Göteborg, Sweden. ISBN 91 85974 66-8

Andreas Werckmeister: “Orgel-Probe.” Quedlinburg 1681.

Andreas Werckmeister: “Erweiterte und verbesserte Orgel-Probe.” Quedlinburg 1698. Facsimile-Nachdrück herausgegeben von Dietz-Rüdiger Moser, Documenta Musicologica, Erste Reihe, XXX, Bärenreiter (1970).

Stephen Bicknell: “Temperament: A Beginner's Guide.” <www.zentao.com/guitar/lesson5/temperament.html> (April 25, 1997; Revised May 14, 1997) (Accessed May 9, 2007).